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FOUNDATION • MEDICAL • IIT-JEE

MATHEMATICS

COMPLEX NUMBER

1 INTRODUCTION

Whenever $\sqrt{x}$ is thought of to give a real value, it has been, till now, insisted that $x \geq 0$. In other words, in the set of real numbers it is not possible to provide for the existence of a value for $\sqrt{x}$ when $x < 0$. To make this possible we extend the number system so as to include and cover yet another class of numbers, called imaginary numbers.

Let us take the quadratic equation $x^2 - 2x + 10 = 0$. The formal solution of this equation is $\frac{2 \pm \sqrt{4-40}}{2}$ i.e., $1 \pm 3\sqrt{-1}$, which is not meaningful in the set of real numbers.

It is therefore, the symbol i, is thought of to possess the following properties:

(i) It combines with itself and with real numbers satisfying the laws of algebra.

(ii) Whenever we come across –1 we may substitute i^2.

In the light of the foregoing the roots of the equation discussed earlier may be taken as $1 + 3i$, $1 - 3i$.

It is taken that 1 is real part and 3(or – 3) is the imaginary part of this complex number $1 + 3i$ or $1 - 3i$ respectively.

It has now to be mentioned that, + symbol standing between 1 and $3i$ does not appear to be meaningful; though the following are true.

$$(x_1 + iy_1) \pm (x_2 + iy_2) = (x_1 \pm x_2) + i(y_1 \pm y_2) \quad \text{.... (i)}$$

i.e., the real parts are added (or subtracted) separately and so in fact the imaginary parts.

Also, $$(x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1) \quad \text{.... (ii)}$$

$$\frac{x_1 + iy_1}{x_2 + iy_2} = \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + \frac{i(x_2y_1 - x_1y_2)}{x_2^2 + y_2^2} \quad \text{.... (iii)}$$

To make these operations really meaningful, a formal extension of the number system is done in this lesson.

Illustration - 1

Question: **If $x = -5 + 2\sqrt{-4}$ find the value of $x^4 + 9x^3 + 35x^2 - x + 4$.**

Solution: $x = -5 + 4i \ (i = \sqrt{-1})$

$x + 5 = 4i$

Squaring, $x^2 + 10x + 25 = -16 \Rightarrow x^2 + 10x + 41 = 0$

Now $x^4 + 9x^3 + 35x^2 - x + 4 = (x^2 + 10x + 41)(x^2 - x + 4) - 160$ and $x^2 + 10x + 41 = 0$

Hence given expression $= 0 - 160 = -160$

2 COMPLEX NUMBERS

A complex number, represented by an expression of the form $x + iy$ (x, y are real), is taken to be an ordered pair (x, y) of two real numbers, combined to form a complex number and an algebra is defined on the set of such numbers, represented by an ordered pair (x, y) to satisfy the following:

(addition) $(x_1,y_1)+(x_2,y_2)=(x_1+x_2,y_1+y_2)$

(subtraction) $(x_1,y_1)-(x_2,y_2)=(x_1-x_2,y_1-y_2)$

(multiplication) $(x_1,y_1)\times(x_2,y_2)=(x_1x_2-y_1y_2,x_1y_2+x_2y_1)$

(division) $(x_1,y_1)\div(x_2,y_2)=\left(\dfrac{x_1x_2+y_1y_2}{x_2^2+y_2^2},\dfrac{x_2y_1+x_1y_2}{x_2^2+y_2^2}\right)$

For any real number $\alpha,\alpha(x, y) = (\alpha x,\alpha y)$ and if $(x, y) = (x', y')$ then it must be $x' = x$; $y' = y$. In other words, the representation of a complex number in the form (x, y) has a uniqueness property; and for a complex number it is not possible to have two different ordered pairs form of representation. In the light of the foregoing it may be said that the two representation (x, y) – in the ordered pair form and $x + iy$ are indistinguishable.

Illustration - 2

Question: **Find the sum and product of the two complex numbers $Z_1 = 2+3i$ and $Z_2 = -1+5i$**

Solution: $Z_1+Z_2 = 2+3i+(-1+5i) = 2-1+8i = 1+8i$

$$Z_1Z_2 = (2+3i)(-1+5i) = -2+15i^2-3i+10i = -17+7i \quad (i^2=-1)$$

Based on the above discussion we are listing a few points:

1. If $z = a + ib$, then real part of z = Re (z) = a and Imaginary part of z = *Im* (z) = b.
2. If Re (z) = 0, the complex number is purely imaginary.
3. If Im (z) = 0, the complex number is real.
4. The complex number $0 = 0 + 0i$ is both purely imaginary and real.
5. Two complex numbers are equal if and only if their real parts and imaginary parts are separately equal i.e. $a + ib = c + id \Leftrightarrow a = c$ and $b = d$.
6. There is no order relation between complex numbers i.e. ($a + ib$) > or < ($c + id$) is a meaningless expression.

Illustration - 3

Question: **Express $\dfrac{1}{(1-\cos\theta+i\sin\theta)}$ in the form $a + ib$.**

Solution: $\dfrac{1}{(1-\cos\theta+i\sin\theta)} = \dfrac{(1-\cos\theta)-i\sin\theta}{(1-\cos\theta+i\sin\theta)(1-\cos\theta-i\sin\theta)}$

$$=\{(1-\cos\theta)-i\sin\theta\}/\{(1-\cos\theta)^2+\sin^2\theta\} = \frac{(1-\cos\theta)-i\sin\theta}{2-2\cos\theta}$$

$$=\frac{1-\cos\theta}{2(1-\cos\theta)}-\frac{i\cdot 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}} = \frac{1}{2}-i\cdot\cot\frac{\theta}{2}$$

3 REPRESENTATION OF A COMPLEX NUMBER

3.1 GEOMETRICAL REPRESENTATION

It is known, from coordinate geometry, that the ordered pair (x, y) represents a point in the Cartesian plane.

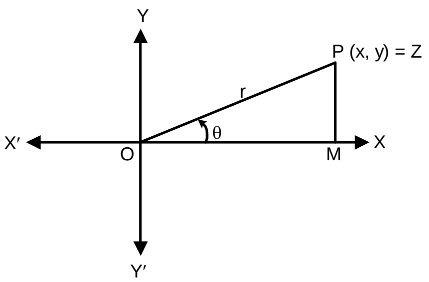
It is now seen that the ordered pair (x, y) taken as Z represents a complex number.

It is therefore, that to every complex number $Z \equiv (x, y)$, one can associate, a point $P \equiv (x, y)$ in the Cartesian plane. The point may be said to be geometrical representation of Z. This association is a bijection – in the mapping language – whereby, this correspondence between Z and P is ONE-ONE and ONTO. It is therefore possible to go over to a point from Z, or reversing the roles, come back to Z from the point.

3.2 ARGAND DIAGRAM

The graphical representation of a complex number $Z = (x, y)$ by a point $P(x, y)$ is called representation in the Argand's Diagram also called Gaussian plane. In this representation, all complex numbers like (2, 0), (3, 0), (–1, 0), (α, 0) with imaginary part 0 will be represented by points on the x-axis. Since the real number α is represented as a complex number (α, 0), all real numbers will get marked on the x-axis. For this reason, the x-axis is called the real axis. Similarly all purely imaginary numbers (with real part 0) like (0, 1), (0, 2), (0, – 3), (0,β) will be marked on the y-axis. Hence the y-axis is also called the imaginary axis in this context. The Cartesian plane (two dimensional plane) is also called the complex plane.

3.3 POLAR REPRESENTATION



Let $P(x, y)$ be any point on the complex plane representing the complex number $z = (x, y)$, with $X'OX$ and $Y'OY$ as the axes of coordinates.

Let $OP = r$ and $\angle XOP = \theta$ (measured in anticlockwise).

Then from ΔOMP, we find that $x = OM = r\cos\theta$ and $y = MP = r\sin\theta$

Thus $z = (x, y) = x + iy = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta)$

where $e^{i\theta} = \cos\theta + i\sin\theta$

$e^{-i\theta} = \cos\theta - i\sin\theta$ by eulers formula

Thus $z = r(\cos\theta + i\sin\theta)$ can be written as

$\boldsymbol{z = re^{i\theta}}$

This form of representation of Z is called the **trigonometric form** or the **polar form** or the **modulus amplitude form**.

When z is written in the form $r(\cos\theta + i\sin\theta)$, r is called the modulus of z and is written as $|z|$; $|z| = r = \sqrt{x^2+y^2}$, a non-negative number. $|z| = 0$ for the only number (0, 0).

Illustration - 4

Question: **Represent the given complex numbers in polar form:**

(i) $\boldsymbol{(1+i\sqrt{3})^2 / 4i(1-i\sqrt{3})}$ **(ii)** $\boldsymbol{\sin\alpha - i\cos\alpha}$ **(α acute)** **(iii)** $\boldsymbol{1+\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}}$

Solution: (i) $i(1-i\sqrt{3}) = i - i^2\sqrt{3} = \sqrt{3} + i$

$$\therefore \frac{(1+i\sqrt{3})^2}{4i\,(1-i\sqrt{3})} = \frac{(1+i\sqrt{3})^2}{4(\sqrt{3}+i)} = \frac{-2+2i\sqrt{3}}{4(\sqrt{3}+i)} = \frac{(-1+i\sqrt{3})\,(\sqrt{3}-i)}{2(\sqrt{3}+i)\,(\sqrt{3}-i)} = \frac{-\sqrt{3}+\sqrt{3}+4i}{2(3+1)} = \frac{i}{2}$$

and $\frac{i}{2} = \frac{1}{2}\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$. Hence $\frac{(1+i\sqrt{3})^2}{4i\,(1-i\sqrt{3})} = \frac{1}{2}\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = \frac{1}{2}\,e^{i\pi/2}$

(ii) Real part > 0; Imaginary part < 0

argument of $\sin\alpha - i\cos\alpha$ is in the nature of a negative acute angle.

$$\therefore \sin\alpha - i\cos\alpha = \cos\left(\alpha - \frac{\pi}{2}\right) + i\sin\left(\alpha - \frac{\pi}{2}\right) = e^{i\left(\alpha - \frac{\pi}{2}\right)}$$

(iii) $1 + \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} = 2\cos^2\frac{\pi}{6} + i\cdot 2\sin\frac{\pi}{6}\cos\frac{\pi}{6}$

$$= 2\cos\frac{\pi}{6}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = 2\cos\frac{\pi}{6}e^{i\pi/6}$$

3.4 VECTOR REPRESENTATION OF A COMPLEX NUMBER

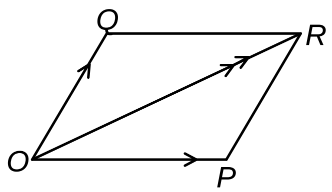
In the Argand's diagram any complex number $Z = x + iy$ can be represented by a point P with coordinates (x, y). The vector $\overline{OP}$ can also be used to represent Z. The length of the vector $\overline{OP}$, (i.e.,) OP is the modulus of Z and the angle θ that OP makes with the positive X-axis is the amplitude of Z.

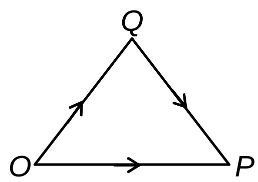
Representation of an algebraic operation on complex numbers

Sum: If two complex numbers Z_1 and Z_2 be represented by the points P and Q or by $\overline{OP}$ and $\overline{OQ}$, then the sum $Z_1 + Z_2$ is represented by R or $\overline{OR}$, where $\overline{OR} = \overline{OP} + \overline{OQ}$ and OR is the diagonal of the parallelogram with OP and OQ as adjacent sides.

Difference: $Z_1 - Z_2$ will be represented by $\overline{QP}$ since $\overline{QP} = \overline{OP} - \overline{OQ}$.

$Z_2 - Z_1$ will be represented by $\overline{PQ}$.

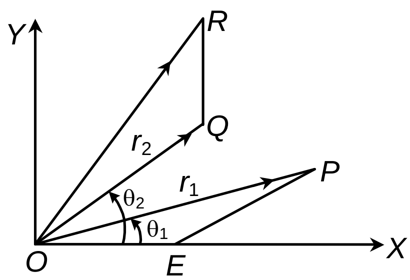




Multiplication: If $Z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$, $Z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$,

then $Z_1 Z_2 = r_1 r_2 \{\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\}$.

If $\overline{OP}$ and $\overline{OQ}$ represent Z_1 and Z_2, construct ΔOQR similar to ΔOEP where $OE = 1$.



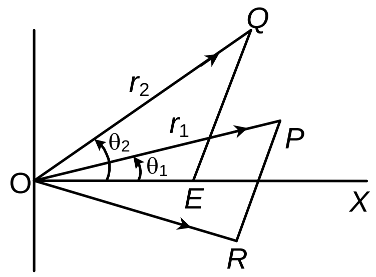
$\angle XOR = \angle XOQ + \angle QOR = \angle XOQ + \angle EOP = \theta_2 + \theta_1$

and $\frac{OR}{OQ} = \frac{OP}{OE}$, $\quad\therefore OR = OP \cdot OQ = r_1 r_2 \quad$ {as $OE = 1$}

Hence $\overline{OR}$ represents the product $Z_1 Z_2$.

Division

$$\frac{Z_1}{Z_2} = \left(\frac{r_1}{r_2}\right)\{\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\}$$



Construct ΔORP similar to ΔOEQ

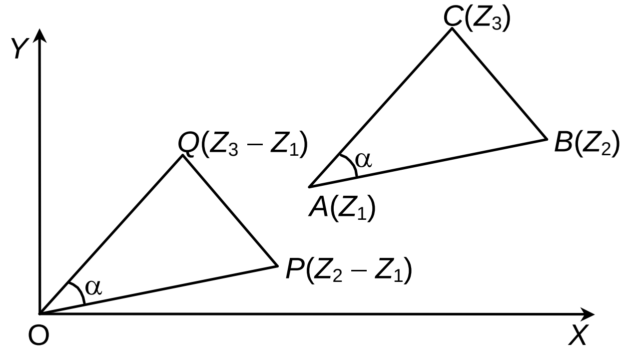
Now $\frac{OR}{OE} = \frac{OP}{OQ} \Rightarrow OR = \frac{r_1}{r_2}$ and $\angle ROX = \angle ROP - \angle EOP = \angle EOQ - \angle EOP = \theta_2 - \theta_1$

$\therefore \angle XOR = \theta_1 - \theta_2$

Hence $\overline{OR}$ represents $\frac{Z_1}{Z_2}$.

Corollary 1: If Z_1, Z_2, Z_3 are the vertices of a triangle ABC described in the counter-clockwise direction, then

$\frac{Z_3 - Z_1}{Z_2 - Z_1} = \frac{CA}{BA}$ (cos α + *i* sin α), where $\alpha = \angle BAC$



Let P and Q be the points representing $Z_2 - Z_1$ and $Z_3 - Z_1$.
Then the triangles POQ and BAC are congruent.

$\therefore \frac{CA}{BA} = \frac{OQ}{OP}$ and $\angle QOP = \angle BAC = \alpha$

Now $\frac{Z_3 - Z_1}{Z_2 - Z_1}$ has modulus $\frac{OQ}{OP} = \frac{CA}{BA}$ and argument $\angle POQ = \alpha$

Hence $\frac{Z_3 - Z_1}{Z_2 - Z_1} = \left(\frac{CA}{BA}\right)(\cos\alpha + i\sin\alpha)$

In particular, if $\alpha = 90°$ and $AB = AC$, then $\frac{Z_3 - Z_1}{Z_2 - Z_1} = i$ or $(Z_3 - Z_1) = i(Z_2 - Z_1)$

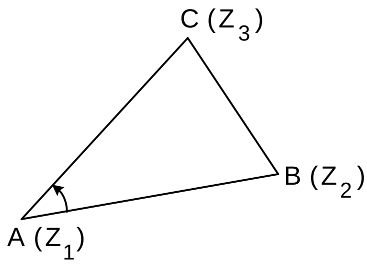
Corollary 2

If Z_1, Z_2, Z_3 are represented by A, B, C, then

$\arg\left(\frac{Z_3 - Z_1}{Z_2 - Z_1}\right) = \angle BAC$

$\arg\left(\frac{Z_2 - Z_3}{Z_1 - Z_3}\right) = \angle ACB$ and

$\arg\left(\frac{Z_1 - Z_2}{Z_3 - Z_2}\right) = \angle CBA$



4 CONJUGATE OF A COMPLEX NUMBER

The complex numbers $z = (a, b) = a + ib$ and $\bar{z} = (a, -b) = a - ib$, where a and b are real numbers, $i = \sqrt{-1}$ and $b \neq 0$ are said to be complex conjugate of each other. (Here the complex conjugate is obtained by just changing the sign of i).

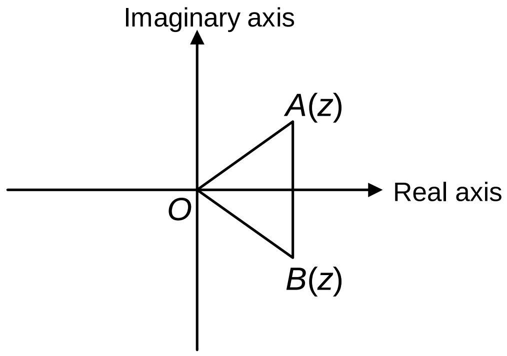
Note that, sum = $(a + ib) + (a - ib) = 2a$ which is real

and product $= (a + ib)(a - ib) = a^2 - (ib)^2 = a^2 - i^2b^2$

$= a^2 - (-1)\, b^2 = a^2 + b^2$ which is real.

4.1 PROPERTIES OF CONJUGATE

- $\overline{(\bar{z})} = z$
- $z = \bar{z} \Leftrightarrow z$ is real
- $z = -\bar{z} \Leftrightarrow z$ is purely imaginary
- $\text{Re}(z) = \text{Re}(\bar{z}) = \dfrac{z + \bar{z}}{2}$
- $\text{Im}(z) = \dfrac{z - \bar{z}}{2i}$
- $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$
- $\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$
- $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\overline{z_1}}{\overline{z_2}} \quad (z_2 \neq 0)$
- $z_1\overline{z_2} + \overline{z_1}z_2 = 2\text{Re}(\overline{z_1}z_2) = 2\text{Re}(z_1\overline{z_2})$
- $\overline{z^n} = (\bar{z})^n$
- If $z = f(z_1)$, then $\bar{z} = f(\overline{z_1})$



5 MODULUS OF A COMPLEX NUMBER

Modulus of a complex number $z = x + iy$ is a real number given by $|z| = \sqrt{x^2 + y^2}$. It is always non-negative and $|z| = 0$ only for $z = 0$ i.e. origin of Argand plane. Geometrically it represents the distance of the point complex number from its origin.

5.1 PROPERTIES OF MODULUS

- $|z| \geq 0 \Rightarrow |z| = 0$ iff $z = 0$ and $|z| > 0$ iff $z \neq 0$.
- $-|z| \leq \text{Re}(z) \leq |z|$ and $-|z| \leq |z|$.
- $|z| = |\bar{z}| = |-z| = |-\bar{z}|$
- $z\bar{z} = |z|^2$
- $|z_1z_2| = |z_1||z_2|$
  In general $|z_1z_2z_3 \ldots\ldots z_n| = |z_1|\,|z_2|\,|z_3| \ldots\ldots |z_n|$
- $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|} \quad (z_2 \neq 0)$
- $|z_1 \pm z_2| \leq |z_1| + |z_2|$
  In particular, if $|z_1 + z_2| = |z_1| + |z_2|$, then origin, z_1 and z_2 are collinear with origin at one of the ends.
- $|z_1 \pm z_2| \geq ||z_1| - |z_2||$
  In particular, if $|z_1 - z_2| = ||z_1| - |z_2||$, then origin, z_1 and z_2 are collinear with origin at one of the ends.
- $|z^n| = |z|^n$
- $||z_1| - |z_2|| \leq |z_1| + |z_2|$

Thus $|z_1| + |z_2|$ is the greatest possible value of $|z_1 + z_2|$ and $|\,|z_1| - |z_2|\,|$ is the least possible value of $|z_1 + z_2|$

- $|z_1 \pm z_2|^2 = (z_1 \pm z_2)\,(\overline{z_1} \pm \overline{z_2}) = |z_1|^2 + |z_2|^2 \pm (z_1\overline{z_2} + \overline{z_1}z_2)$ or $|z_1|^2 + |z_2|^2 \pm 2\text{Re}(z_1\overline{z_2})$
- $z_1\overline{z_2} + \overline{z_1}z_2 = 2|z_1||z_2|\cos(\theta_1 - \theta_2)$ where $\theta_1 = \arg(z_1)$ and $\theta_2 = \arg(z_2)$.
- $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 \Leftrightarrow \dfrac{z_1}{z_2}$ is purely imaginary.
- $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\left\{|z_1|^2 + |z_2|^2\right\}$
- $|az_1 - bz_2|^2 + |bz_1 + az_2|^2 = (a^2 + b^2)\,(|z_1|^2 + |z_2|^2)$ where $a, b \in \boldsymbol{R}$.
- Unimodular : i.e., unit modulus

  If z is unimodular then $|z| = 1$. A unimodular complex number can always be expressed as $\cos\theta + i\sin\theta$, $\theta \in \boldsymbol{R}$.

  Note: $\dfrac{z}{|z|}$ is always a unimodular complex number if $z \neq 0$.

Some of the proofs are given as:

- $|Z_1 Z_2| = |Z_1| \times |Z_2|$

Proof:

Let $Z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $Z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

Then $Z_1 Z_2 = r_1 r_2\,\{\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\} = r(\cos\theta + i\sin\theta)$,

where $r = r_1 r_2$ and $\theta = \theta_1 + \theta_2$.

$\therefore |Z_1 Z_2| = r = r_1 r_2 = |Z_1| \times |Z_2|$

- $|Z_1 Z_2 \ldots Z_n| = |Z_1| \times |Z_2| \times |Z_3| \times \ldots \times |Z_n|$

  Proof follows by writing $Z_1 Z_2 \ldots Z_n$ as the product of $Z_1 Z_2 \ldots Z_{n-1}$ and Z_n and applying property (1) repeatedly.
- $|Z^n| = |Z|^n$

  Proof follows if we take $Z_1 = Z_2 = Z_3 = \ldots = Z_n$
- $\left|\dfrac{Z_1}{Z_2}\right| = \dfrac{|Z_1|}{|Z_2|}$

Proof:

Let $Z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $Z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

Now $\dfrac{Z_1}{Z_2} = \dfrac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} = \dfrac{r_1}{r_2}(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 - i\sin\theta_2)$

$$\left(\text{since}\ \frac{1}{\cos\theta_2 + i\sin\theta_2} = \cos\theta_2 - i\sin\theta_2\right)$$

$$= \left(\frac{r_1}{r_2}\right)\{(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)\}$$

$$= \left(\frac{r_1}{r_2}\right)\{\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\}$$

Hence $\left|\dfrac{Z_1}{Z_2}\right| = \dfrac{r_1}{r_2} = \dfrac{|Z_1|}{|Z_2|}$

- First triangle inequality $|Z_1| + |Z_2| \ge |Z_1 + Z_2|$

Proof:

$$|Z_1 + Z_2| = |r_1(\cos\theta_1 + i\sin\theta_1) + r_2(\cos\theta_2 + i\sin\theta_2)|$$
$$= |(r_1\cos\theta_1 + r_2\cos\theta_2) + i(r_1\sin\theta_1 + r_2\sin\theta_2)|$$
$$= \sqrt{(r_1\cos\theta_1 + r_2\cos\theta_2)^2 + (r_1\sin\theta_1 + r_2\sin\theta_2)^2} = \sqrt{r_1^2 + r_2^2 + 2r_1r_2\cos(\theta_1 - \theta_2)}$$
$$\le \sqrt{r_1^2 + r_2^2 + 2r_1r_2}, \text{ since } \cos(\theta_1 - \theta_2) \le 1$$
$$\therefore |Z_1 + Z_2| \le \sqrt{(r_1 + r_2)^2}$$

or $|Z_1 + Z_2| \le r_1 + r_2$. Thus $|Z_1 + Z_2| \le |Z_1| + |Z_2|$.

Note: Equality occurs only when $\theta_1 = \theta_2$ i.e. when Z_1 and Z_2 have the same amplitude.

- Second triangle inequality
  $|Z_1 - Z_2| \ge |Z_1| \sim |Z_2|$

Proof

$$Z_1 - Z_2 = r_1\cos\theta_1 - r_2\cos\theta_2 + i(r_1\sin\theta_1 - r_2\sin\theta_2)$$
$$\therefore |Z_1 - Z_2| = \sqrt{(r_1\cos\theta_1 - r_2\cos\theta_2)^2 + (r_1\sin\theta_1 - r_2\sin\theta_2)^2} = \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1 - \theta_2)}$$
$$\ge \sqrt{r_1^2 + r_2^2 - 2r_1r_2}, \text{ since } \cos(\theta_1 - \theta_2) \le 1$$
$$\therefore |Z_1 - Z_2| \ge \sqrt{(r_1 - r_2)^2} = |r_1 - r_2|$$
$$|Z_1 - Z_2| \ge r_1 \sim r_2 = |Z_1| \sim |Z_2|.$$

- $|\bar{Z}| = |Z|$

Proof:

$|Z| = \sqrt{x^2 + y^2}$ if $Z = x + iy$

Then $\bar{Z} = x - iy$

$$\therefore |\bar{Z}| = \sqrt{x^2 + (-y)^2} = \sqrt{x^2 + y^2}$$
$$\therefore |\bar{Z}| = |Z|$$

Illustration - 5

Question: **If $|z - 2 + i| \le 2$ then find the greatest and least value of $|z|$.**

Solution: Given that

$|z - 2 + i| \le 2$ …(i)

$\because$ $|z - 2 + i| \ge ||z| - |2 - i||$

$\therefore$ $|z - 2 + i| \ge ||z| - \sqrt{5}|$ …(ii)

From (i) and (ii)

$||z| - \sqrt{5}| \le |z - 2 + i| \le 2$

$\therefore$ $||z| - \sqrt{5}| \le 2$

$\Rightarrow$ $-2 \le |z| - \sqrt{5} \le 2$

$\Rightarrow$ $\sqrt{5} - 2 \le |z| \le \sqrt{5} + 2$

Hence greatest value of $|z|$ is $\sqrt{5} + 2$ and least value of $|z|$ is $\sqrt{5} - 2$.

Illustration - 6

Question: **If $\left|Z+\frac{1}{Z}\right|=a$, where Z is a complex number and a is a positive real number, then find the greatest $|Z|$ and least $|Z|$.**

Solution: Let us first find greatest $|Z|$

If $|Z|$ is greatest, $\frac{1}{|Z|}$ is least and hence $|Z| > \frac{1}{|Z|}$

Write $a = \left|Z+\frac{1}{Z}\right| = \left|Z-\left(-\frac{1}{Z}\right)\right| \geq |Z|-\frac{1}{|Z|}$

This gives $|Z|^2 - a|Z| - 1 \leq 0$; and hence $|Z|$ lies between the roots of the equation $|Z|^2 - a|Z| - 1 = 0$

Roots are $\frac{a \pm \sqrt{a^2+4}}{2}$ and hence $\frac{a-\sqrt{a^2+4}}{2} \leq |Z| \leq \frac{a+\sqrt{a^2+4}}{2}$ ... (i)

It is known that $|Z| \geq 0$ while $\frac{a-\sqrt{a^2+4}}{2}$ is < 0 and hence (i) gets modified as

$0 \leq |Z| \leq \frac{a+\sqrt{a^2+4}}{2}$

and thus the greatest value of $|Z|$ is $\frac{a+\sqrt{a^2+4}}{2}$

Now for the least $|Z|$.

In this case $\frac{1}{|Z|}$ is greatest and hence $\frac{1}{|Z|}-|Z|>0$

write $a = \left|Z+\frac{1}{Z}\right| = \left|\frac{1}{Z}-(-Z)\right| \geq \frac{1}{|Z|}-|Z|$

This gives $|Z|^2 + a|Z| - 1 \geq 0$ and this is possible for all $|Z|$ lying outside the roots of $|Z|^2 + a|Z| - 1 = 0$

Roots are $\frac{-a \pm \sqrt{a^2+4}}{2}$; and of these $\frac{-a-\sqrt{a^2+4}}{2}$ is negative, hence |Z| cannot be les than this negative value.

Therefore $|Z| \geq \frac{-a+\sqrt{a^2+4}}{2}$ and this gives the least $|Z|$ value.

Illustration - 7

Question: **If Z_1 and Z_2 be two complex numbers such that $\left|\frac{Z_1-2Z_2}{2-Z_1\bar{Z}_2}\right|=1$ and $|Z_2| \neq 1$. What is the value of $|Z_1|$?**

Solution: $|Z_1 - 2Z_2| = |2 - Z_1\bar{Z}_2|$

$\therefore |Z_1 - 2Z_2|^2 = |2 - Z_1\bar{Z}_2|^2$

$\therefore (Z_1 - 2Z_2)(\bar{Z}_1 - 2\bar{Z}_2) = (2 - Z_1\bar{Z}_2)(2 - \bar{Z}_1 Z_2)$

$\therefore Z_1\bar{Z}_1 - 2\bar{Z}_1 Z_2 - 2Z_1\bar{Z}_2 + 4Z_2\bar{Z}_2 = 4 - 2Z_1\bar{Z}_2 - 2\bar{Z}_1 Z_2 + Z_1\bar{Z}_1 Z_2\bar{Z}_2$

$\therefore Z_1\bar{Z}_1 + 4Z_2\bar{Z}_2 - 4 - Z_1\bar{Z}_1 Z_2\bar{Z}_2 = 0$

$|Z_1|^2 + 4|Z_2|^2 - |Z_1|^2|Z_2|^2 - 4 = 0$ i.e. $\left(|Z_1|^2 - 4\right)\left(|Z_2|^2 - 1\right) = 0$

Since $|Z_2| \neq 1$ it is that $|Z_1|^2 = 4$ i.e. $|Z_1| = 2$

6 ARGUMENT OF A COMPLEX NUMBERS

If $z = x + iy = r(\cos\theta + i\sin\theta)$, where $r = \sqrt{x^2 + y^2}$.

θ is called the argument of Z or amplitude of Z. Since $x = r\cos\theta$ and $y = r\sin\theta$, θ is such that $\cos\theta = \frac{x}{\sqrt{x^2+y^2}}$ and $\sin\theta = \frac{y}{\sqrt{x^2+y^2}}$. Since there can be many values of θ satisfying these conditions, by convention, θ such that $-\pi < \theta \le \pi$ is defined as the principal argument of Z and is denoted by arg Z. The argument of a complex number $a + ib$ is given by α, $\pi-\alpha$, $-\pi + \alpha$, or $-\alpha$ if $a + ib$ is in first, second, third or fourth quadrant respectively, where $\alpha = \tan^{-1}\left|\frac{b}{a}\right|$.

For example

- $Z = 1 + i = (1, 1)$ and is marked by point $P(1, 1)$ lies in first quadrant.
  $\therefore$ $|Z| = \sqrt{2}$ and arg $Z = \pi/4$.
- If $Z = 1 - i = (1, -1)$, then P lies in the fourth quadrant and $|Z| = \sqrt{2}$ and arg $Z = -\pi/4$.
- If $Z = -1 + i = (-1, 1)$, then P lies in the second quadrant and arg $Z = \frac{3\pi}{4}$.
- If $Z = -1 - i$, arg $Z = -\frac{3\pi}{4}$.
- Argument of all positive real numbers like 1, 2, 3, $\frac{1}{2}$, … is 0 since they are marked on the positive x-axis. Argument of all negative real numbers like – 1, – 2, – 3, … is π since they are marked on OX′. Argument of purely imaginary numbers like i, $2i$, $3i$, … is $\frac{\pi}{2}$ since these are marked on the positive y-axis. Argument of purely imaginary numbers like $-i$, $-2i$, $-3i$, … is $-\frac{\pi}{2}$.

Illustration - 8

Question: **Among the complex numbers z which satisfies $|z - 25i| \le 15$, find the complex numbers z having**

(i) least positive argument **(ii) maximum positive argument**

(iii) least modulus **(iv) maximum modulus**

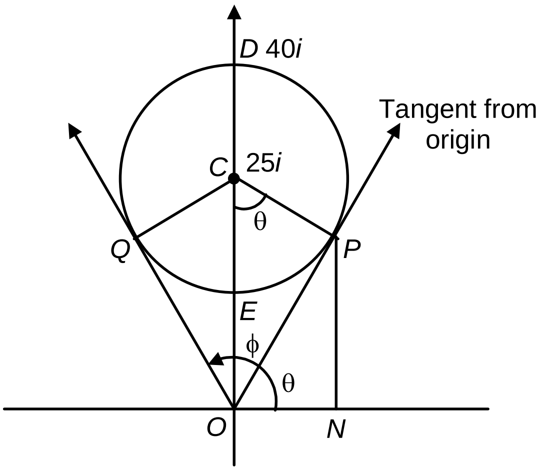
Solution: The complex numbers z satisfying the condition $|z - 25i| \le 15$ are represented by the points inside and on the circle of radius 15 and centre at the point $C(0, 25)$.

The complex number having least positive argument and maximum positive arguments in this region are the points of contact of tangents drawn from origin to the circle.

Here θ = least positive argument

and ϕ = maximum positive argument

$\therefore$ In ΔOCP, $OP = \sqrt{(OC)^2 - (CP)^2} = \sqrt{(25)^2 - (15)^2} = 20$



and $\sin\theta = \dfrac{OP}{OC} = \dfrac{20}{25} = \dfrac{4}{5}$

$\therefore$ $\tan\theta = \dfrac{4}{3}$

$\Rightarrow$ $\theta = \tan^{-1}\left(\dfrac{4}{3}\right)$

Thus, complex number at P has modulus 20 and argument $\theta = \tan^{-1}\left(\dfrac{4}{3}\right)$

$\therefore$ $Z_P = 20(\cos\theta + i\sin\theta) = 20\left(\dfrac{3}{5} + i\dfrac{4}{5}\right)$

$\therefore$ $Z_P = 12 + 16i$

Similarly $Z_Q = -12 + 16i$

From the figure, E is the point with least modulus and D is the point with maximum modulus.

Hence, $Z_E = \overrightarrow{OE} = \overrightarrow{OC} - \overrightarrow{EC} = 25i - 15i = 10i$

and $Z_D = \overrightarrow{OD} = \overrightarrow{OC} + \overrightarrow{CD} = 25i + 15i = 40i$

6.1 PROPERTIES OF ARGUMENTS

- Arg (z_1z_2) = Arg (z_1) + Arg (z_2) + $2k\pi$ ($k = 0$ or 1 or – 1)
  In general Arg $(z_1z_2z_3 \ldots z_n)$ = Arg (z_1) + Arg (z_2) + Arg (z_3) + …… + Arg (z_n) + $2k\pi$ (where $k \in \boldsymbol{I}$)
- Arg $\left(\dfrac{z_1}{z_2}\right)$ = Arg z_1 – Arg z_2 + $2k\pi$ ($k = 0$ or 1 or –1)
- Arg $\left(\dfrac{z}{\bar{z}}\right) = 2$ Arg z + $2k\pi$ ($k = 0$ or 1 or –1)
- Arg (z^n) = n Arg z + $2k\pi$ ($k = 0$ or 1 or –1)
- If Arg $\left(\dfrac{z_2}{z_1}\right) = \theta$, then Arg $\left(\dfrac{z_1}{z_2}\right) = 2k\pi - \theta$ where $k \in \boldsymbol{I}$.
- Arg $\bar{z}$ = – Arg z
- If arg $(z) = 0$ $\Rightarrow$ z is real.

Note: Proper value of k must be chosen so, that R.H.S. of (i), (ii), (iii), (iv) lies in $(-\pi, \pi]$.

All the above formulae are written on the basic of principal argument.

Illustration - 9

Question: **Let z, z_0 be two complex numbers. It is given that $|z| = 1$ and the numbers z, z_0, $z\bar{z}_0$, 1 and 0 are represented in an argand diagram by the points P, P_0, Q, A and the origin O respectively. Show that the triangles POP_0 and AOQ are congruent. Hence, or otherwise, prove that $|z - z_0| = |z\bar{z}_0 - 1|$.**

Solution: Given $OA = 1$ and $|z| = 1$

$\therefore \quad OP = |z - 0| = |z| = 1$

$\therefore \quad OP = OA$

$OP_0 = |z_0 - 0| = |z_0|$

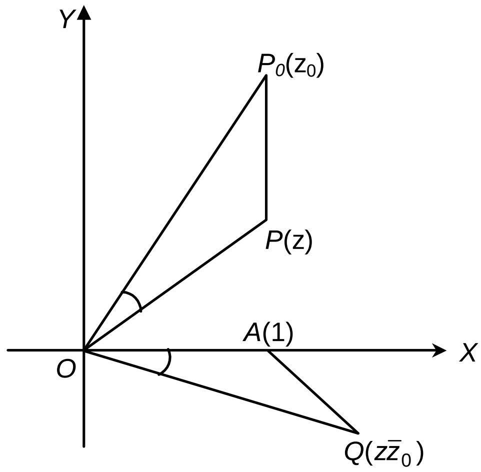
and $\quad OQ = |z\bar{z}_0 - 0| = |z\bar{z}_0|$

$= |z|\,|\bar{z}_0| = 1\,|z_0| = |z_0|$

$\therefore \quad OP_0 = OQ$

and

$$\angle P_0OP = \arg\left(\frac{z_0 - 0}{z - 0}\right) = \arg\left(\frac{z_0}{z}\right)$$



$$= \arg\left(\frac{\bar{z}z_0}{z\bar{z}}\right) = \arg\left(\frac{\bar{z}z_0}{|z|^2}\right) = \arg\left(\frac{\bar{z}z_0}{1}\right) = -\arg(\overline{\bar{z}z_0}) = -\arg(z\bar{z}_0) = \arg\left(\frac{1}{z\bar{z}_0}\right)$$

$$= \arg\left(\frac{1-0}{z\bar{z}_0 - 0}\right) = \angle AOQ$$

Thus, the triangles POP_0 and AOQ are congruent.

Also, $\quad PP_0 = AQ$

$\Rightarrow \quad |z - z_0| = |z\bar{z}_0 - 1|$

Illustration - 10

Question: **If arg ($z^{1/3}$) = $\frac{1}{2}$ arg ($z^2 + \bar{z}z^{1/3}$), then find the value of |z|.**

Solution: We have $\arg(z^{1/3}) = \frac{1}{2}\arg(z^2 + \bar{z}z^{1/3})$

$\Rightarrow \quad 2\arg(z^{1/3}) = \arg(z^2 + \bar{z}z^{1/3})$

$\Rightarrow \quad \arg(z^{2/3}) = \arg(z^2 + \bar{z}z^{1/3})$ (By prop.)

$\Rightarrow \quad \arg(z^2 + \bar{z}z^{1/3}) - \arg(z^{2/3}) = 0$

$\Rightarrow \quad \arg\left(\frac{z^2 + \bar{z}z^{1/3}}{z^{2/3}}\right) = 0$ (By prop.)

$\Rightarrow \quad \arg\left(z^{4/3} + \frac{\bar{z}}{z^{1/3}}\right) = 0$

$\Rightarrow \quad z^{4/3} + \frac{\bar{z}}{z^{1/3}}$ is real. $\Rightarrow \quad \text{Im}\left(z^{4/3} + \frac{\bar{z}}{z^{1/3}}\right) = 0$

$$\Rightarrow \quad \frac{\left(z^{4/3} + \frac{\bar{z}}{z^{1/3}}\right) - \overline{\left(z^{4/3} + \frac{\bar{z}}{z^{1/3}}\right)}}{2i} = 0$$

$$\Rightarrow \quad z^{4/3} + \frac{\bar{z}}{z^{1/3}} = (\bar{z})^{4/3} + \frac{(\overline{\bar{z}})}{(\bar{z})^{1/3}}$$

$$\Rightarrow \quad z^{4/3} + \frac{(\bar{z})\,(\bar{z})^{1/3}}{|z|^{2/3}} = (\bar{z})^{4/3} + \frac{z\,(z)^{1/3}}{|z|^{2/3}} \qquad [\because z^{1/3}(\bar{z})^{1/3} = (z\bar{z})^{1/3} = |z|^{2/3}]$$

$$\Rightarrow \quad z^{4/3} - (\bar{z})^{4/3} - \frac{1}{|z|^{2/3}}((z)^{4/3} - (\bar{z})^{4/3}) = 0$$

$$\{z^{4/3} - (\bar{z})^{4/3}\}\left[1 - \frac{1}{|z|^{2/3}}\right] = 0 \qquad \therefore |z|^{2/3} = 1 \ (\because z \neq \bar{z})$$

Therefore, $\quad |z| = 1$

Illustration - 11

Question: **If $|Z| \le 1$; $|W| \le 1$; show that $|Z - W|^2 \le (|Z| - |W|)^2 + (\arg Z - \arg W)^2$.**

Solution: Let $Z = |Z|(\cos\theta + i\sin\theta)$ *and* $W = |W|(\cos\phi + i\sin\phi)$

$$|Z - W|^2 = (|Z|\cos\theta - |W|\cos\phi)^2 + (|Z|\sin\theta - |W|\sin\phi)^2$$

$$= |Z|^2(\cos^2\theta + \sin^2\theta) + |W|^2(\cos^2\phi + \sin^2\phi) - 2|Z||W|(\cos\theta\cos\phi + \sin\theta\sin\phi)$$

$$= |Z|^2 + |W|^2 - 2|Z||W|\cos(\theta - \phi)$$

$$= (|Z| - |W|)^2 + 2|Z||W|(1 - \cos(\theta - \phi))$$

$$= (|Z| - |W|)^2 + 4|Z||W|\sin^2\left(\frac{\theta - \phi}{2}\right)$$

$$\le (|Z| - |W|)^2 + (\theta - \phi)^2 \qquad \text{as } |Z| \le 1, |W| \le 1 \text{ and } \sin^2(\theta - \phi) \le \left(\frac{\theta - \phi}{2}\right)^2$$

PROFICIENCY TEST–I

The following questions deal with the basic concepts of this section. Answer the following briefly. Go to the next section only if your score is at least 80%. Do not consult the Study Material while attempting the questions.

1. If $(a + 2b) - i(2a - b) = 2i - 6$ then find a and b.

2. Find the value of $\sum_{k=1}^{4n+7} i^k$.

3. If $a = \frac{1+i}{\sqrt{2}}$ then prove that the value of a^{1929} is also equal to $\frac{1+i}{\sqrt{2}}$.

4. If $z_1 = 2 - 3i$ and $z_2 = 2 + 7i$ find $|z_1 - z_2|$ and arg $(z_1 - z_2)$.

5. What is the polar form of $z = 1 - i\sqrt{3}$.

6. If $a + ib = \frac{(2+3i)^2}{2+i}$, find a and b.

7. Find the value of $i^{13} + i^{14} + i^{15} + i^{16}$.

8. Find the least non-zero positive integer n such that $\left(\frac{1+i}{1-i}\right)^n = 1$.

9. If $X + iY = (x + iy)^{1/3}$ then prove that $4(X^2 - Y^2) = \frac{x}{X} + \frac{y}{Y}$.

10. If $|z_1| = |z_2| = |z_3| .... = |z_n| = 1$, then prove that

$$|z_1 + z_2 + z_3 + ... + z_n| = \left|\frac{1}{z_1} + \frac{1}{z_2} + ... + \frac{1}{z_n}\right|$$

ANSWERS TO PROFICIENCY TEST–I

1. $a = -2,\ b = -2.$

2. -1

4. $|z_1 - z_2| = 10$ and $\text{Arg}\,(z_1 - z_2) = -\pi/2$

5. $z = 2e^{i(-\pi/3)}$

6. $a = \frac{2}{5},\ b = \frac{29}{5}$

7. 0

8. $n = 4$

7 De MOIVRE'S THEOREM

For any rational number n, the value or one of the values of $(\cos\theta + \sin\theta)^n$ is $(\cos n\theta + \sin n\theta)$. The following may also be noted:

(a) $(\cos\theta + i\sin\theta)^{-n} = (\cos n\theta - \sin n\theta) = (\cos\theta - i\sin\theta)^n$

(b) $(\cos\theta + i\sin\theta)^n = (\cos n\theta + \sin n\theta) = (\cos\theta - i\sin\theta)^{-n}$

(c) If $x + \frac{1}{x} = 2\cos\theta$; and if the equation is solved for x then $x = \cos\theta + i\sin\theta = e^{i\theta}$; then

$$\frac{1}{x} = \cos\theta - i\sin\theta = e^{-i\theta}$$

or, $x = \cos\theta - i\sin\theta = e^{-i\theta}$; then $\frac{1}{x} = \cos\theta + i\sin\theta = e^{i\theta}$

Illustration - 12

Question: **If $2\cos\theta = x + \frac{1}{x}$ and $2\cos\phi = y + \frac{1}{y}$, prove the following**

(i) $x^m y^n + \frac{1}{x^m y^n} = 2\cos(m\theta + n\phi)$

(ii) $\frac{x^m}{y^n} + \frac{y^n}{x^m} = 2\cos(m\theta - n\phi)$

Solution: (i) Given $x + \frac{1}{x} = 2\cos\theta \Rightarrow x^2 - 2x\cos\theta + 1 = 0$. Solving this, $x = \cos\theta \pm i\sin\theta$

In fact if $x = \cos\theta + i\sin\theta$; $\frac{1}{x} = \cos\theta - i\sin\theta$. It may be noted also that $x + \frac{1}{x} = 2\cos\theta$ is symmetrical w.r.t. $\frac{1}{x}$ and hence if one root is the value for x, the other root is $\frac{1}{x}$ and vice-versa.

Similarly, given that $2\cos\theta = y + \frac{1}{y}$, we have $y = \cos\phi + i\sin\phi$

$$\therefore x^m = (\cos\theta + i\sin\theta)^m = \cos m\theta + i\sin m\theta; \textit{ and}$$

$$y^n = (\cos\phi + i\sin\phi)^n = \cos n\phi + i\sin n\phi$$

$$x^m y^n = (\cos m\theta + i\sin m\theta)(\cos n\phi + i\sin n\phi)$$

$$= \cos(m\theta + n\phi) + i\sin(m\theta + n\phi)$$

and $\frac{1}{x^m y^n} = \cos(m\theta + n\phi) - i\sin(m\theta + n\phi)$

Adding we get $x^m y^n + \frac{1}{x^m y^n} = 2\cos(m\theta + n\phi)$

(ii) By similar reasoning $\frac{x^m}{y^n} + \frac{y^n}{x^m} = 2\cos(m\theta - n\phi)$

Illustration - 13

Question: **If n be a positive integer, prove that**

$$(1+i)^{2n}+(1-i)^{2n}=\begin{cases}0 & \text{if n be odd}\\ 2^{n+1} & \text{if } \frac{n}{2} \text{ be even}\\ -2^{n+1} & \text{if } \frac{n}{2} \text{ be odd}\end{cases}$$

Solution:

$$(1+i)^{2n}=2^n\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)^{2n}=2^n\left(\cos\frac{n\pi}{2}+i\sin\frac{n\pi}{2}\right)$$

$$(1-i)^{2n}=2^n\left(\cos\frac{\pi}{4}-i\sin\frac{\pi}{4}\right)^{2n}=2^n\left(\cos\frac{n\pi}{2}-i\sin\frac{n\pi}{2}\right)$$

$$\therefore\ (1+i)^{2n}+(1-i)^{2n}=2^n\left(\cos\frac{n\pi}{2}+i\sin\frac{n\pi}{2}+\cos\frac{n\pi}{2}-i\sin\frac{n\pi}{2}\right)$$

$$=2^{n+1}\cos\left(\frac{n\pi}{2}\right)$$

If n be odd $= 2m + 1$, then RHS = 2 cos $(2m + 1)\ \frac{\pi}{2}$

$= 0$

If n be even and $\frac{n}{2}$ also even so that $n = 4k$, then RHS = $2^{n+1}\cos(2k\pi)=2^{n+1}$

else RHS = $2^{n+1}\cos\left(\frac{n\pi}{2}\right)=-2^{n+1}$

8 ROOTS OF UNITY

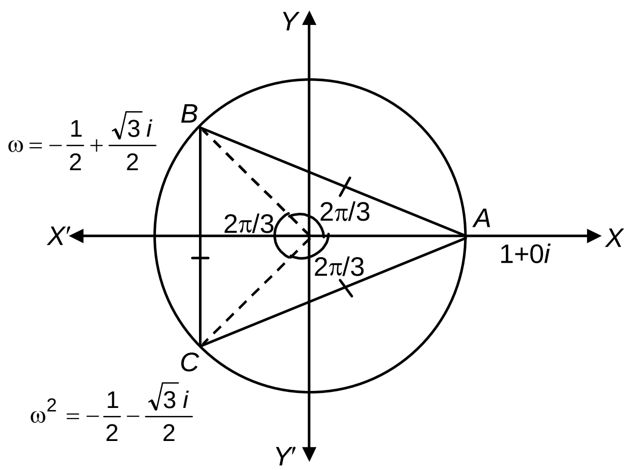
8.1 CUBE ROOTS OF UNITY

Consider the cubic (3rd degree) equation

$$x^3=1=\cos 0+i\sin 0=\cos 2k\pi+i\sin 2k\pi$$

$$\therefore x=\sqrt[3]{1}=(\cos 2k\pi+i\sin 2k\pi)^{1/3}$$

$$=\cos\left(\frac{2k\pi}{3}\right)+i\sin\left(\frac{2k\pi}{3}\right)$$



To get the three roots of the cubic equation we give
$k = 0$, giving the real root, $\cos 0 + i \sin 0 = 1$

$k = 1$, giving one imaginary root, $\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = \omega$

$k = 2$, giving the other imaginary root, $\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = \omega^2$

It is said that 1, ω, ω^2 are the three cubic roots of unity satisfying

(i) $1 + \omega + \omega^2 = 0$

(ii) $\omega^3 = 1$

(iii) 1, ω, ω^2 are represented respectively by points A, B, C lying on the unit circle $|Z| = 1$ and forming the corners of an equilateral triangle of sides $\sqrt{3}$.

Illustration - 14

Question: **If α, β, γ are roots of $x^3 - 3x^2 + 3x + 7 = 0$ (and ω is cube roots of unity), then find the value of $\frac{\alpha-1}{\beta-1} + \frac{\beta-1}{\gamma-1} + \frac{\gamma-1}{\alpha-1}$.**

Solution: We have $x^3 - 3x^2 + 3x + 7 = 0$

$\therefore \quad (x-1)^3 + 8 = 0 \quad \therefore \quad (x-1)^3 = (-2)^3$

$\Rightarrow \quad \left(\frac{x-1}{-2}\right)^3 = 1 \quad \Rightarrow \quad \frac{x-1}{-2} = (1)^{1/3} = 1, \omega, \omega^2$ (cube roots of unity)

$\therefore \quad x = -1, 1 - 2\omega, 1 - 2\omega^2$

Here $\alpha = -1, \beta = 1 - 2\omega, \gamma = 1 - 2\omega^2$

$\therefore \quad \alpha - 1 = -2, \beta - 1 = -2\omega, \gamma - 1 = -2\omega^2$.

Then $\frac{\alpha-1}{\beta-1} + \frac{\beta-1}{\gamma-1} + \frac{\gamma-1}{\alpha-1} = \left(\frac{-2}{-2\omega}\right) + \left(\frac{-2\omega}{-2\omega^2}\right) + \left(\frac{-2\omega^2}{-2}\right)$

$$= \frac{1}{\omega} + \frac{1}{\omega} + \omega^2$$

$$= \omega^2 + \omega^2 + \omega^2 = 3\omega^2$$

8.2 Some useful results

$(x^3 + y^3) = (x + y)(x + \omega y)(x + \omega^2 y)$

$(x^3 - y^3) = (x - y)(x - \omega y)(x - \omega^2 y)$

$(x^3 + y^3 + z^3 - 3xyz) = (x + y + z)(x + \omega y + \omega^2 z)(x + \omega^2 y + \omega z)$

Illustration - 15

Question: **If $a = 1 + \frac{x^3}{3!} + \frac{x^6}{6!} + \ldots\ldots\ldots\infty$**

$$b = x + \frac{x^4}{4!} + \frac{x^7}{7i} + \ldots\ldots\ldots\infty$$

$$c = \frac{x^2}{2!} + \frac{x^5}{5!} + \frac{x^8}{8!} + \ldots\ldots\infty$$

Then find the value of $a^3 + b^3 + c^3 - 3abc$

Solution: $a^3 + b^3 + c^3 - 3abc = (a + b + c)(a + b\omega + c\omega^2)(a + b\omega^2 + c\omega)$

$$= e^x e^{\omega x} e^{\omega^2 x}$$

$$= e^{x(1+\omega+\omega^2)}$$

$$= e^0 = 1$$

8.3 n^{th} ROOTS OF UNITY

More generally the nth degree equation $x^n = 1$ has 'n' nth roots of unity given by

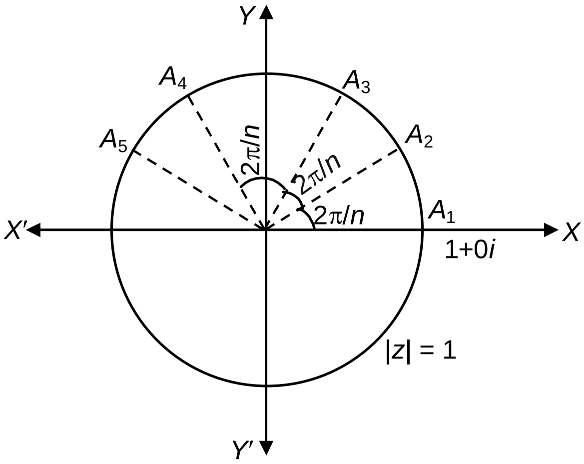
$$\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}, \cos\frac{4\pi}{n} + i\sin\frac{4\pi}{n}, \ldots, \cos\frac{2(n-1)\pi}{n} + i\sin\frac{2(n-1)\pi}{n}$$

i.e. $1, \alpha, \alpha^2 \ldots \alpha^{n-1}$ satisfying

(i) $1 + \alpha + \alpha^2 + \ldots + \alpha^{n-1} = 0$

(ii) $\alpha^n = 1$

(iii) $1, \alpha, \ldots, \alpha^{n-1}$ represent n points in the Argand plane situated on the unit circle $|Z| = 1$ and forming the corners of a regular n sides polygon.



As sum of nth roots of unity = 0

$$\Rightarrow \quad \sum_{k=0}^{n-1} \alpha^k = 0$$

$$\Rightarrow \quad \sum_{k=0}^{n-1}\left(\cos\left(\frac{2k\pi}{n}\right) + i\left(\sin\frac{2k\pi}{n}\right)\right)$$

$$\Rightarrow \quad \sum_{k=0}^{n-1}\cos\left(\frac{2k\pi}{n}\right) = 0 \quad \text{and} \quad \sum_{k=0}^{n-1}\sin\left(\frac{2k\pi}{n}\right) = 0$$

More general equation like $x^n = a + ib$ can be solved by using this method.

First write $a + ib = r[\cos\theta + i\sin\theta] = r[\cos(\theta + 2k\pi) + i\sin(\theta + 2k\pi)]$ and hence the n n^{th} roots of $x^n = a + ib$ are $\sqrt[n]{r}\left(\cos\frac{\theta + 2k\pi}{n} + i\sin\frac{\theta + 2k\pi}{n}\right); k = 0, 1, 2 \ldots, (n-1)$

Illustration - 16

Question: **Solve $2\sqrt{2}\,x^5 = (\sqrt{3} - 1) + i(\sqrt{3} + 1)$.**

Solution: $2\sqrt{2}\,x^5 = (\sqrt{3} - 1) + i(\sqrt{3} + 1)$

$$x^5 = \left(\frac{\sqrt{3} - 1}{2\sqrt{2}}\right) + i\left(\frac{\sqrt{3} + 1}{2\sqrt{2}}\right)$$

$$x^5 = \cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}$$

$$\left(\because \cos\frac{5\pi}{12} = \cos 75^\circ = \frac{\sqrt{3} - 1}{2\sqrt{2}}; \text{and} \quad \sin\frac{5\pi}{12} = \sin 75^\circ = \frac{\sqrt{3} + 1}{2\sqrt{2}}\right)$$

$$= \cos\left(2k\pi + \frac{5\pi}{12}\right) + i\sin\left(2k\pi + \frac{5\pi}{12}\right)$$

$\therefore$ The 5 roots of the given equation are

$$x = \cos\left(\frac{2k\pi + \frac{5\pi}{12}}{5}\right) + i\sin\left(\frac{2k\pi + \frac{5\pi}{12}}{5}\right) (k = 0, 1, 2, 3, 4)$$

Illustration - 17

Question: **If 1, α, α^2 .... α^{n-1} are nth roots of unity then prove that**

(a) $(1-\alpha)(1-\alpha^2)\ldots\ldots\ldots(1-\alpha^{n-1}) = n$

(b) $\sin\frac{\pi}{n}\sin\frac{2\pi}{n}\sin\frac{3\pi}{n}\ldots\ldots\sin\frac{(n-1)\pi}{n} = \frac{n}{2^{n-1}}, n \ge 2$

Solution: If 1, α, α^2 .... α^{n-1} are roots of $x^n = 1$

$\Rightarrow \quad x^n - 1 = (x-1)(x-\alpha)(x-\alpha^2)\ldots\ldots(x-\alpha^{n-1})$

$\Rightarrow \quad (x-\alpha)(x-\alpha^2)\ldots\ldots(x-\alpha^{n-1}) = \frac{x^n-1}{x-1} = 1 + x + x^2 + \ldots\ldots + x^{n-1}$

Put $x = 1$ $\quad \Rightarrow \quad (1-\alpha)(1-\alpha^2)\ldots\ldots(1-\alpha^{n-1}) = n$

Also $\alpha^k = \frac{2k\pi}{n} \quad \Rightarrow \quad |1-\alpha^k| = \left|2\sin\frac{k\pi}{n}\right|$

Taking modulus of the first result

$\Rightarrow \quad |1-\alpha|\,|1-\alpha^2|\ldots\ldots|1-\alpha^{n-1}| = |n|$

$\Rightarrow \quad \left(2\sin\frac{\pi}{n}\right)\left(2\sin\frac{2\pi}{n}\right)\ldots\ldots\left(2\sin\frac{(n-1)\pi}{n}\right) = n$

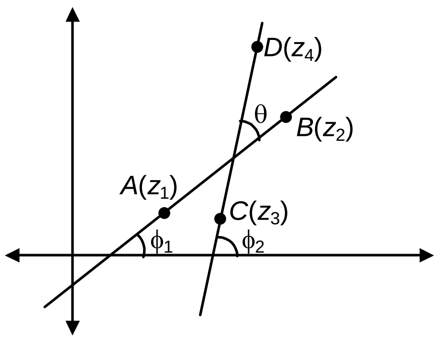
$\Rightarrow \quad \sin\frac{\pi}{n}\sin\frac{2\pi}{n}\ldots\ldots\sin(n-1)\frac{\pi}{n} = \frac{n}{2^{n-1}}$

9 ROTATION THEOREM

9.1 CONI METHOD

This method gives the angle between two intersecting lines.

Let z_1, z_2, z_3 and z_4 be complex numbers representing points A, B, C, D respectively.



Then $\overrightarrow{AB} = z_2 - z_1$

$\overrightarrow{CD} = z_4 - z_3$

Let arg $\overrightarrow{AB} = \phi_1$ and arg $\overrightarrow{CD} = \phi_2$ then angle of intersection

$$\theta = \phi_2 - \phi_1 = \arg\overrightarrow{CD} - \arg\overrightarrow{AB} = \arg(z_4 - z_3) - \arg(z_2 - z_1) = \arg\left(\frac{z_4 - z_3}{z_2 - z_1}\right)$$

(i) If $\theta = 0$ or $\pm\pi$, then $\left(\frac{z_4 - z_3}{z_2 - z_1}\right)$ is real. Points are collinear as the two lines coincide.

It follows that if $\left(\frac{z_4 - z_3}{z_2 - z_1}\right)$ is real, points are collinear.

(ii) If $\theta = \pm\frac{\pi}{2}$, then $\left(\frac{z_4 - z_3}{z_2 - z_1}\right)$ is purely imaginary. It follows that if $\left(\frac{z_4 - z_3}{z_2 - z_1}\right)$ is purely imaginary then line joining z_1, z_2 is perpendicular to the line joining z_3, z_4.

(iii) Hence angle between the lines passing through z_2 and z_3 and intersecting at z_1 is given by

$$\arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right) = \theta$$

also $z = |z| e^{i\theta}$

$$\Rightarrow \quad \frac{z_3 - z_1}{z_2 - z_1} = \left|\frac{z_3 - z_1}{z_2 - z_1}\right| e^{i\theta}$$

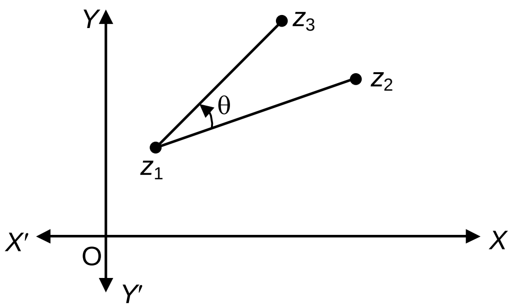


Illustration - 18

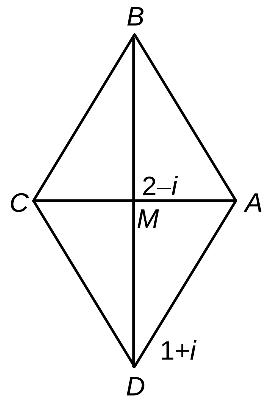
Question: ***ABCD* is a rhombus. Its diagonals *AC* and *BD* intersect at *M* such that *BD* = 2*AC*. If the points *D* and *M* represent the complex number 1 + *i* and 2 – *i* respectively, find the complex number(s) representing *A*.**

Solution: Let *A* be *z*. The position *MA* can be obtained by rotating *MD* anticlockwise through an angle $\frac{\pi}{2}$; simultaneously length gets halved.

$$\therefore\ z - (2 - i) = \frac{1}{2}\left((1 + i) - (2 - i)\right)e^{i\pi/2}$$

$$= \frac{1}{2}(-2 - i) = -1 - \frac{1}{2}i$$

$$z = -1 - \frac{1}{2}i + 2 - i = 1 - \frac{3i}{2}$$

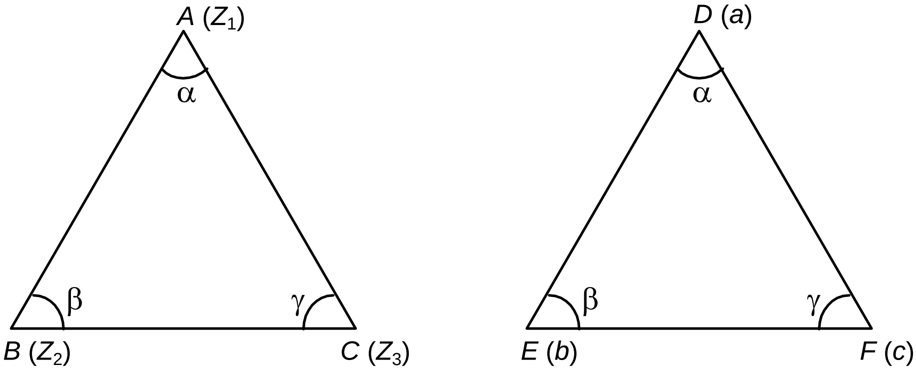


Another position of *A* corresponds to *A* and *C* getting interchanged and in that the complex number of *A* is $1 + \frac{1}{2}i + 2 - i = 3 - \frac{1}{2}i$

$\therefore$ The complex number of *A* is either $1 - \frac{3i}{2}$ or $3 - \frac{1}{2}i$

Illustration - 19

Question: **Show that the triangles whose vertices are Z_1, Z_2, Z_3 and *a*, *b*, *c* (Z_1, Z_2, Z_3 and *a*, *b*, *c* are complex) are similar if** $\begin{vmatrix} Z_1 & a & 1 \\ Z_2 & b & 1 \\ Z_3 & c & 1 \end{vmatrix} = 0$.



Solution: The two triangles are similar if

$$\frac{AB}{DE} = \frac{BC}{EF} \text{ and } \angle ABC = \angle DEF = \beta \text{ (say)}$$

$$\therefore \frac{Z_1 - Z_2}{Z_3 - Z_2} = \frac{AB}{BC}(\cos\beta + i\sin\beta)$$

Similarly $\frac{a-b}{c-b} = \frac{DE}{EF}(\cos\beta + i\sin\beta)$

$$\therefore \frac{Z_1 - Z_2}{Z_3 - Z_2} = \frac{a-b}{c-b} \quad \text{i.e.} \quad \begin{vmatrix} Z_1 - Z_2 & a-b \\ Z_3 - Z_2 & c-b \end{vmatrix} = 0$$

i.e., $\begin{vmatrix} Z_1 - Z_2 & a-b & 0 \\ Z_2 & b & 1 \\ Z_3 - Z_2 & c-b & 0 \end{vmatrix} = 0$

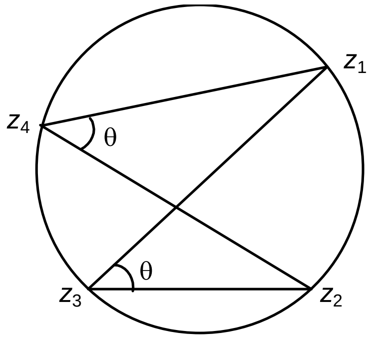
i.e. $\begin{vmatrix} Z_1 & a & 1 \\ Z_2 & b & 1 \\ Z_3 & c & 1 \end{vmatrix} = 0$ adding R_2 to R_1 and R_2 to R_3

9.2 CONDITION FOR FOUR POINTS TO BE CONCYCLIC

Four points z_1, z_2, z_3 and z_4 in the Argand plane are concyclic if and only if

$$\arg\left(\frac{z_1 - z_3}{z_2 - z_3}\right) = \arg\left(\frac{z_1 - z_4}{z_2 - z_4}\right) = \theta \text{ (say)}$$

Applying coni method, we get



$$\frac{z_1 - z_3}{z_2 - z_3} = \left|\frac{z_1 - z_3}{z_2 - z_3}\right| e^{i\theta}$$

$$\frac{z_1 - z_4}{z_2 - z_4} = \left|\frac{z_1 - z_4}{z_2 - z_4}\right| e^{i\theta}$$

Solving the above two equations to eliminate θ we get

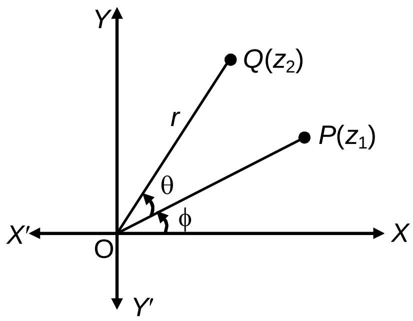
$$\frac{z_1 - z_3}{z_2 - z_3} \cdot \frac{z_2 - z_4}{z_1 - z_4} = \left|\frac{z_1 - z_3}{z_2 - z_3}\,\frac{z_2 - z_4}{z_1 - z_4}\right|$$

This is possible only if the expression on left hand side is real (it may be positive or negative, depending upon whether the points are considered in cyclic order or not).

9.3 COMPLEX NUMBER AS A ROTATING ARROW IN THE ARGAND PLANE

(i) If a complex number z_1 is rotated in anticlockwise sense by an angle θ and let z_2 be its new position, then $z_1 = re^{i\phi}$ and $z_2 = re^{i(\theta+\phi)}$ (as $|z_1| = |z_2| = r$)

$\Rightarrow$ $\boldsymbol{z_2 = z_1 e^{i\theta}}$



Clearly multiplication of z with $e^{i\theta}$ rotates vector $\overrightarrow{OP}$ through an angle θ in anticlockwise sense. Similarly multiplication of z with $e^{-i\theta}$ will rotate vector $\overrightarrow{OP}$ in clockwise sense.

(ii) If z_1, z_2, z_3 are the affixes of three points A, B, C such that $AC = AB$ and $\angle CAB = \theta$. Then $\overrightarrow{AC} = z_3 - z_1$ will be obtained by rotating $\overrightarrow{AB} = z_2 - z_1$ through an angle θ in anticlockwise sense and therefore

$$(z_3 - z_1) = (z_2 - z_1)e^{i\theta}$$

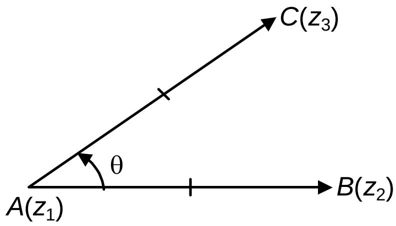
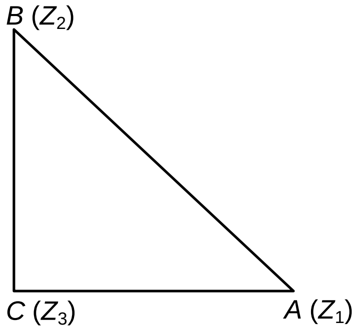


Illustration - 20

Question: **Complex numbers Z_1, Z_2, Z_3 are the vertices A, B and C respectively of an isosceles right angled triangle with $\angle C = 90°$. Show that $(Z_1 - Z_2)^2 = 2(Z_1 - Z_3)(Z_3 - Z_2)$ (OR) equivalently $Z_1^2 + Z_2^2 + 2Z_3^2 = 2Z_1Z_3 + 2Z_2Z_3$.**

Solution: It is seen that when CA is turned anticlockwise through an angle $90°$, the position of CB is obtained. Lengthwise $CA = CB$ since the triangle is isosceles.



$$\therefore Z_2 - Z_3 = (Z_1 - Z_3)\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$$

Squaring $(Z_2 - Z_3)^2 + (Z_1 - Z_3)^2 = 0$

i.. $\quad Z_1^2 + Z_2^2 + 2Z_3^2 = 2Z_1Z_3 + 2Z_2Z_3$

which is the second result.

To get the first from the second, we have,

$$Z_1^2 + Z_2^2 = 2Z_1Z_3 + 2Z_2Z_3 - 2Z_3^2$$

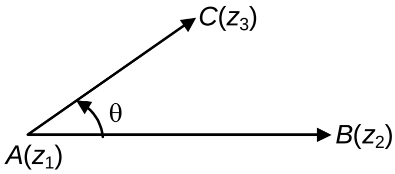
$$Z_1^2 + Z_2^2 - 2Z_1Z_2 = 2Z_1Z_3 + 2Z_2Z_3 - 2Z_3^2 - 2Z_1Z_2$$

i.e. $\quad (Z_1 - Z_2)^2 = 2(Z_1 - Z_3)(Z_3 - Z_2)$

which is the desired from of the result.

(iii) In the above case if $AB \neq AC$, then we consider the rotation of unit vectors as

$$\frac{z_3 - z_1}{|z_3 - z_1|} = \frac{z_2 - z_1}{|z_2 - z_1|}e^{i\theta}.$$

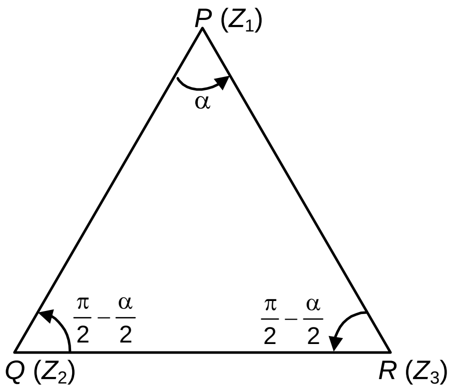


This concept has also been explained in terms of coni method earlier.

Illustration - 21

Question: **The points P, Q and R represent the complex numbers Z_1, Z_2 and Z_3 respectively and the angles of the triangle PQR at Q and R are both $\frac{\pi}{2} - \frac{\alpha}{2}$, Prove that**

$$(Z_3 - Z_2)^2 = 4(Z_3 - Z_1)(Z_1 - Z_2)\sin^2\left(\frac{\alpha}{2}\right).$$



Solution: QP is obtained from QR by a rotation counter clockwise through an angle $\frac{\pi}{2} - \frac{\alpha}{2}$; of course the length PQ is different from the length of QR

$$\therefore Z_1 - Z_2 = \frac{PQ}{QR}(Z_3 - Z_2)\left\{\cos\left(\frac{\pi}{2} - \frac{\alpha}{2}\right) + i\sin\left(\frac{\pi}{2} - \frac{\alpha}{2}\right)\right\}$$

Similarly

$$\therefore Z_1 - Z_3 = \frac{PR}{QR}(Z_2 - Z_3)\left\{\cos\left(\frac{\pi}{2} - \frac{\alpha}{2}\right) - i\sin\left(\frac{\pi}{2} - \frac{\alpha}{2}\right)\right\}$$

Multiplying the two

$$(Z_1 - Z_2)(Z_1 - Z_3) = \frac{PQ \cdot PR}{QR^2}(Z_3 - Z_2)(Z_2 - Z_3)\left\{\cos^2\left(\frac{\pi}{2} - \frac{\alpha}{2}\right) + \sin^2\left(\frac{\pi}{2} - \frac{\alpha}{2}\right)\right\}$$

Now $\dfrac{QR}{\sin\alpha} = \dfrac{PQ}{\cos\frac{\alpha}{2}} = \dfrac{PR}{\cos\frac{\alpha}{2}}$

$$\therefore \frac{PQ \cdot PR}{QR^2} = \frac{\cos^2\frac{\alpha}{2}}{4\sin^2\frac{\alpha}{2}\cos^2\frac{\alpha}{2}}$$

$$\therefore (Z_1 - Z_2)(Z_1 - Z_3)4\sin^2\frac{\alpha}{2} = (Z_3 - Z_2)(Z_2 - Z_3)$$

i.e. $(Z_3 - Z_2)^2 = 4(Z_3 - Z_1)(Z_1 - Z_2)\sin^2\dfrac{\alpha}{2}$.

10 THEORY OF EQUATIONS WITH COMPLEX COEFFICIENTS

An n^{th} degree equation with complex coefficients a_n, a_{n-1}, ……, a_0 is given as

$$a_n z^n + a_{n-1}z^{n-1} + ... + a_1 z + a_0 = 0$$

It has n roots say α_1, α_2, ……..α_n, and

$$\sum \alpha_1 = -\frac{a_{n-1}}{a_n}$$

$$\sum \alpha_1\alpha_2 = +\frac{a_{n-2}}{a_n}$$

$$\alpha_1\alpha_2........\alpha_n = (-1)^n \frac{a_0}{a_n}$$

In case of quadratic equations with complex coefficients having non-zero imaginary part, discriminant has no role for existence of roots.

Illustration - 22

Question: **The roots z_1, z_2, z_3 of the equation $x^3 + 3ax^2 + 3bx + c = 0$, in which a, b, c are complex numbers, correspond to the points A, B, C on the Gaussian plane. Find the centroid of the triangle ABC and show that it will be equilateral if $a^2 = b$.**

Solution: Since, z_1, z_2, z_3 are the roots of $x^3 + 3ax^2 + 3bx + c = 0$

We have $z_1 + z_2 + z_3 = -3a$ or $\dfrac{z_1 + z_2 + z_3}{3} = -a$

and $z_1 z_2 + z_2 z_3 + z_3 z_1 = 3b$

Hence, the centroid of the triangle ABC is the point with affix – a.

Now the triangle will be equilateral if $z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1$

$\Rightarrow$ $(z_1 + z_2 + z_3)^2 = 3(z_1 z_2 + z_2 z_3 + z_3 z_1)$ $\Rightarrow$ $(-3a)^2 = 3(3b)$

Therefore, the condition is $a^2 = b$.

Illustration – 23

Question: **Find the value of |Z| from the equation $2Z^3 - 3Z^2 - 18iZ + 27i = 0$.**

Solution: $2Z^3 - 3Z^2 - 18iZ + 27i = 0$

$Z^2(2Z-3) - 9i(2Z-3) = 0$

$(2Z-3)(Z^2 - 9i) = 0$

$\therefore \quad 2Z - 3 = 0 \Rightarrow |Z| = 3/2$ or $Z^2 = 9i \Rightarrow |Z| = 3$

11 LOGARITHM OF A COMPLEX NUMBER

Let $\log_e(x+iy) = \alpha + i\beta$ …(i)

suppose $x + iy = r(\cos\theta + i\sin\theta)$

$= re^{i\theta}$ …(ii)

then $x = r\cos\theta,\ y = r\sin\theta$

so that $r = \sqrt{(x^2+y^2)}$ and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

from (ii) we have $\log_e(x+iy) = \log_e(re^{i\theta}) = \log_e r + \log_e e^{i\theta}$

$= \log_e r + i\theta$

$= \log_e \sqrt{(x^2+y^2)} + i\tan^{-1}\left(\frac{y}{x}\right)$

or $\log_e(z) = \log_e|z| + i\ \text{amp}\ z$

so, the general value is $\log(z) = \log_e(z) + 2n\pi i$ $(-\pi < \text{amp}(z) < \pi)$.

Illustration - 24

Question: **If $\sin(\log i^i) = a + ib$, find a and b. Hence, find $\cos(\log i^i)$.**

Solution: $a + ib = \sin(\log i^i) = \sin(i\log i)$

$= \sin(i(\log|i| + i\ \text{amp}\,i))$

$= \sin(i(\log 1 + i\pi/2))$

$= \sin(i(0 + i\pi/2))$

$= \sin(-\pi/2) = -1$

$\therefore \quad a = -1, b = 0$

$\therefore \quad \sin(\log i^i) = -1$

now $\cos(\log i^i) = \sqrt{\{1 - \sin^2(\log i^i)\}} = \sqrt{1-1} = 0$

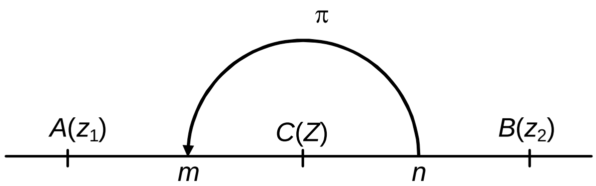
12 SECTION FORMULA

Let z_1 and z_2 represents any two complex number representing the points A and B respectively in the Argand plane. Let C be the point dividing line AB internally in ratio $m : n$ i.e. $\frac{AC}{BC} = m : n$ and let the complex number associated with point C be z.

Then let us rotate the line BC about C so that it becomes parallel to CA. Then corresponding equation after rotation will be

$$\frac{z_1 - z}{z_2 - z} = \frac{|z_1 - z|}{|z_2 - z|} e^{i\pi} = \frac{m}{n} \ (-1)$$

$$\Rightarrow \qquad z = \frac{nz_1 + mz_2}{m+n}$$



Thus

1. If Z_1, Z_2 are divided at P in the ratio $m : n$, then P has the complex number $\dfrac{mZ_2 + nZ_1}{m+n}$.

   Particularly the mid point of the join of Z_1 and Z_2 is $\dfrac{Z_1 + Z_2}{2}$

2. If Z_1, Z_2, Z_3 be three points A, B, C forming a triangle ABC; then the centroid G of triangle ABC has an associated complex number $\dfrac{Z_1 + Z_2 + Z_3}{3}$.

Illustration - 25

Question: **If the vertices of a triangle ABC are represented by Z_1, Z_2 and Z_3 respectively; then prove that**

(i) the centroid is $\dfrac{Z_1 + Z_2 + Z_3}{3}$

(ii) the orthocentre is $\dfrac{(a \sec A) Z_1 + (b \sec B) Z_2 + (c \sec C) Z_3}{a \sec A + b \sec B + c \sec C}$

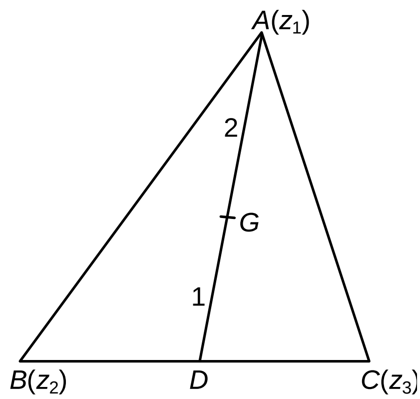
or $\dfrac{(\tan A) z_1 + (\tan B) z_2 + (\tan C) z_3}{\tan A + \tan B + \tan C}$

(iii) the circumcentre is $\dfrac{(\sin 2A) Z_1 + (\sin 2B) Z_2 + (\sin 2C) Z_3}{\sin 2A + \sin 2B + \sin 2C}$

Solution: (i) Mid point D of BC is $\dfrac{Z_2 + Z_3}{2}$ and the point G. on AD, and dividing AD in the ratio 2 : 1 is

$$\frac{2\left(\dfrac{Z_2 + Z_3}{2}\right) + Z_1}{2+1}$$

i.e. $\left(\dfrac{Z_2 + Z_3 + Z_1}{3}\right)$



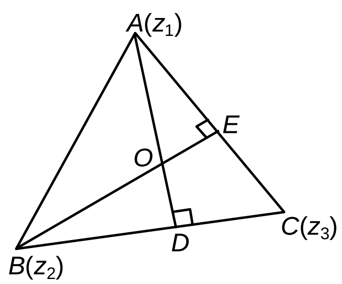
Symmetry in Z_1, Z_2, Z_3 of this result indicates that this point G. lies as well on the other two medians also.

$\therefore$ The medians are concurrent at G, the centroid the associated complex of which is $\dfrac{Z_1 + Z_2 + Z_3}{3}$

(ii) Orthocentre

Let the two altitudes AD and BE intersect at O

Now, $\dfrac{BD}{DC} = \dfrac{c \cos B}{b \cos C} = \dfrac{c \sec C}{b \sec B}$



The point D, dividing BC in the ratio $\dfrac{BD}{DC}$ has a complex number

$\dfrac{(c \sec C) Z_3 + (b \sec B) Z_2}{b \sec B + c \sec C}$; Again; $\dfrac{AO}{OD} = \dfrac{\text{Area of } \Delta \text{ABO}}{\text{Area of } \Delta \text{OBD}}$ (triangles of the same altitude)

$$= \frac{1}{2} AB \cdot BO \sin \angle ABE \Big/ \frac{1}{2} BO \cdot BD \sin \angle DBE$$

$$= c \cos A/(c \cos B \cdot \cos C) = \frac{a \cos A}{a \cos B \cos C} = \frac{b \cos C + c \cos B}{\cos B \cos C} \frac{1}{a \sec A}$$

$$= \frac{b \sec B + c \sec C}{a \sec A}$$

$\therefore$ The point O, dividing AD, in the ratio $\frac{AO}{OD}$ has a complex number

$$\frac{AO \text{ (complex number of } D) + OD \text{ (complex number of } A)}{AO + OD}$$

$$= \frac{(b \sec B + c \sec C)\left(\dfrac{b \sec B \cdot Z_2 + c \sec C \cdot Z_3}{b \sec B + c \sec C}\right) + a \sec A \cdot Z_1}{b \sec B + c \sec C + a \sec A}$$

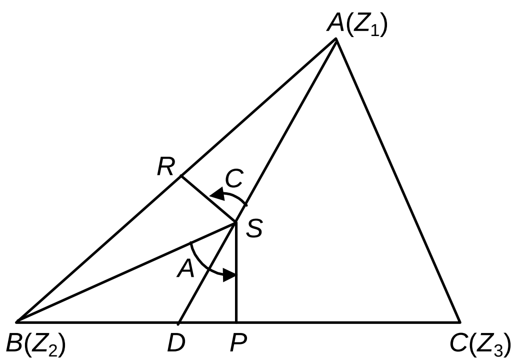
$$= \frac{(a \sec A)Z_1 + (b \sec B)Z_2 + (c \sec C)Z_3}{a \sec A + b \sec B + c \sec C}$$

The symmetry of this result in a, b, c and A, B, C indicates that O lies on the third altitude also. Hence O, the orthocentre, is $\frac{Z_1 a \sec A + Z_2 b \sec B + Z_3 c \sec C}{a \sec A + b \sec B + c \sec C}$

To prove the other result substituting $a = 2R \sin A$, $b = 2R \sin B$ and $c = 2R \sin C$ in the above result.

$$\frac{Z_1 \tan A + Z_2 \tan B + Z_3 \tan C}{\tan A + \tan B + \tan C}$$

(iii) Circum-centre



Let S be the point of intersection of perpendicular bisectors of BC and AB. S lies on the third perpendicular bisector also

Let AS produced meet BC in D. Now.

$$\frac{BD}{DC} = \frac{\text{area of } \Delta ABD}{\text{area of } \Delta ACD} \text{ (Triangles of the same altitude)}$$

$$= \frac{AB \cdot AD \cdot \sin \angle BAD}{AC \cdot AD \cdot \sin \angle CAD} = \frac{c \sin (90^\circ - C)}{b \sin (90^\circ - B)}$$

$$= \frac{\sin 2C}{\sin 2B} \quad \ldots \text{(i)}$$

$\therefore$ D is represented by the complex number $\frac{(\sin 2C)Z_3 + (\sin 2B)Z_2}{\sin 2B + \sin 2C}$

$$\frac{AS}{SD} = \frac{\text{area of } \Delta ASB}{\text{area of } \Delta BSD} = \frac{AS \cdot BS \cdot \sin 2C}{BS \cdot BD \sin (90^\circ - A)}$$

$$= \frac{R \sin 2C}{BD \cos A} \quad \ldots \text{(ii)}$$

From (i), $\dfrac{BD}{\sin 2C} = \dfrac{DC}{\sin 2B} = \dfrac{BD + DC}{\sin 2B + \sin 2C} = \dfrac{a}{\sin 2B + \sin 2C}$

Substituting in (ii)

$$\frac{AS}{SD} = \frac{R \sin 2C}{\dfrac{a \sin 2C}{\sin 2B + \sin 2C} \cdot \cos A}$$

$$= \frac{R \sin 2C}{\dfrac{2R \sin A \cos A \sin 2C}{\sin 2B + \sin 2C}} = \frac{\sin 2B + \sin 2C}{\sin 2A}$$

$\therefore$ S is represented by

$$\frac{(\sin 2A)Z_1 + (\sin 2B + \sin 2C)\left(\dfrac{\sin 2C \cdot Z_3 + \sin 2B \cdot Z_2}{\sin 2B + \sin 2C}\right)}{\sin 2A + \sin 2B + \sin 2C}$$

i.e. $\dfrac{Z_1 \sin 2A + Z_2 \sin 2B + Z_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}$

13 LOCUS IN AN ARGAND PLANE

It has been pointed that there is a bijective correspondence between a complex number $Z \equiv (x, y)$ and a point $P(x, y)$ in the complex plane (or) Argand's diagram.

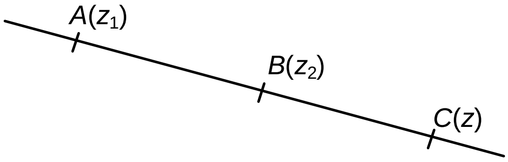
Coordinate Geometry theory gives us the concept of a locus as a curve, every point $P(x, y)$ on which satisfies a relation between x and y termed as the equation to a curve.

But $P(x, y)$ is also equivalent to $Z = (x, y)$; and hence this relation between x and y–representing the equation – can also be put in the form of a condition on Z.

To cite an example, $x^2 + y^2 = 1$, expressed in terms of Z, is $|Z| = 1$; and it is said that the condition $|Z| = 1$, being satisfied by all points Z at units distance from (0, 0), represents a circle with centre at (0, 0) and radius = 1. We therefore assert that any condition imposed on Z, automatically places a restriction on the possible locations in the Argand's diagram of the point P representing Z; and hence all such points lie on a curve. Such a curve traced in the Argand's diagram by $P \equiv Z$, because of a condition imposed on Z, is termed as **Locus in an Argand's diagram**

13.1 STRAIGHT LINE

Equation of straight line passing through points. A and B represented by complex numbers z_1 and z_2 is



Let us take $C(z)$ is the general point on line then

$$\arg\left(\frac{z - z_1}{z_2 - z_1}\right) = 0 \text{ or } \pi$$

$$\Rightarrow \quad \frac{z - z_1}{z_2 - z_1} = \frac{\bar{z} - \bar{z}_1}{\bar{z}_2 - \bar{z}_1}$$

It can also be represented in the following form

$$\begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0$$

The general form of straight line is

$$a\bar{z} + \bar{a}z + b = 0$$

where a is a complex number and b is real.

Slope of a line

Let equation of line be $z\bar{a} + \bar{z}a + b = 0$

Replacing z by x + iy, we get,

$$(a+\bar{a})x + i(\bar{a}-a)y + b = 0$$

Its real slope is = $\dfrac{a+\bar{a}}{i(\bar{a}-a)} = \dfrac{-\text{Re}(a)}{\text{Im}(a)}$

Its complex slope is $-\dfrac{a}{\bar{a}} = -\dfrac{\text{coeff of } \bar{z}}{\text{coeff of } z}$

Equation of the line parallel to $a\bar{z} + a\bar{z} + b$ is $\bar{a}z + a\bar{z} + \lambda = 0$ (where λ is a real number) and that of line perpendicular to it is $z\bar{a} - \bar{z}a + i\lambda = 0$.

Ray

- arg Z = θ is a ray (or a straight line) from the origin and pointed in such a direction that any point Z situated on the lie has an argument θ
- arg $(z - \alpha) = \theta$ is a ray (or a straight line) from the point α and pointed in such a direction that the join of α to Z is inclined at an angle θ to the positive direction of the real axis (x-axis)

Perpendicular bisector

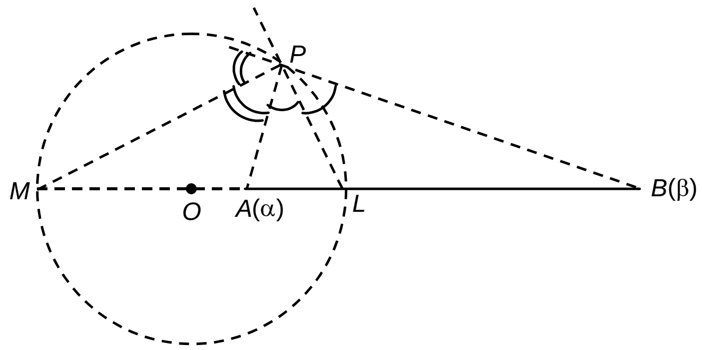
- $|Z - \alpha| = |Z - \beta|$ represents the perpendicular bisector of the join of the two points $\alpha \equiv (p, q)$ and $\beta \equiv (r, s)$.
- Perpendicular distance of a point z_0 from line $\bar{a}z + a\bar{z} + b = 0$ is $= \dfrac{|\bar{a}z_0 + a\bar{z}_0 + b|}{2|a|}$.

13.2 CIRCLE

- $|Z| = r$ is a circle, centre (0, 0) and radius r.
- $|Z - \alpha| = r$ (α, complex) is a circle, centre at $\alpha \equiv (p, q)$ and radius = r since $|Z - \alpha|$ represents the absolute distance of Z from α.
- $|Z - \alpha| = k\,|Z - \beta|$ (k real > 0, $\neq$ 1) is the circle any point P on which, with reference to the points $A(\alpha)$ and $B(\beta)$, satisfies the condition $\dfrac{AP}{PB} = k\ (k \neq 1)$

  Let us take, for exactness, $0 < k < 1$. Let L and M divide the join of $A(\alpha)$; $B(\beta)$ internally at L and externally at M in the ratio k, so that $\dfrac{AL}{LB} = \dfrac{MA}{MB} = k < 1$

  Draw the circle with LM as diameter. Any point P on this circle will satisfy the requirement $\dfrac{AP}{PB} = k$. The locus of the point $P(Z)$ satisfying the condition $\dfrac{|Z-\alpha|}{|Z-\beta|} = k \neq 1$ is this circle on LM as diameter; and is called the **Appolonius Circle** of A and B with respect to the ratio k. The circle meets AB at L and M and these two points, being points on the circle, trivially satisfy the condition to be satisfied by any point P to lie on the circle. In fact the choice of L and M have been made to satisfy this requirement. It may be also pointed, as a property, that PL and PM bisect $\angle APB$ internally and externally.



- $\arg\left(\dfrac{Z-Z_1}{Z-Z_2}\right)=0$ is a straight line – that part of the segment of the line through Z_1 and Z_2 which is outside the segment joining Z_1 and Z_2
- $\arg\left(\dfrac{Z-Z_1}{Z-Z_2}\right)=\pi$ represent the line segment joining Z_1 and Z_2.

  In fact the condition $\arg\left(\dfrac{Z_1-Z_2}{Z_1-Z_3}\right)$ = 0 or π is the condition for Z_1, Z_2 and Z_3 to be collinear.
- $\arg\left(\dfrac{Z-Z_1}{Z-Z_2}\right)=\theta\neq 0\neq\pi$. The equation $\arg\left(\dfrac{Z-Z_1}{Z-Z_2}\right)=\theta$ geometrically expresses the fact that the join of Z_1 and Z_1 subtends at Z, the angle θ. Hence the condition represents the segment of a circle described on the join if Z_1 and Z_2 as a chord and containing at any point $P(Z)$ on the segment the angle θ. If $0<\theta<\pi/2$, the segment is a major segment. If $\pi/2<\theta<\pi$, the segment is a minor segment. If $\theta=\pi/2$ the locus is the semi-circle on the join of Z_1 and Z_1 the circle being appropriately chosen.

  It has already been pointed out that every point can be taken to be represented by a complex number Z. Thus just as in Coordinate Geometry where we have for every point a pair of numbers (its coordinates), in complex number theory every point has an associate complex number, of which, the point is but a geometrical representation.

Illustration - 26

Question: **Show that the equation $Z\bar{Z}+a\bar{Z}+\bar{a}Z+b^2=0$ (*b* is real) is the complex from of the equation to a circle.**

Solution:

$Z\bar{Z}+a\bar{Z}+\bar{a}Z+b^2=0$

$\therefore\quad Z\bar{Z}+a\bar{Z}+\bar{a}Z+a\bar{a}=a\bar{a}-b^2$

i.e. $(Z+a)(\bar{Z}+\bar{a})=|a|^2-b^2$ i.e. $|Z+a|^2=r^2$ where $r^2=|a|^2-b^2$

The last equation represents a circle with centre at $-a$ (complex) and radius $r=\sqrt{|a|^2-b^2}$ and for the circle to be real, we need the condition $|a|^2>b^2$.

Illustration - 27

Question: **Examine what locus is represented by $|Z-a|^2+|Z-b|^2=k$ (where *k* is real).**

Solution:

$|Z-a|^2=(Z-a)(Z-\bar{a})=Z\bar{Z}+a\bar{a}-(Z\bar{a}+\bar{Z}a)$

$=|Z|^2+|a|^2-2\text{Re}(Z\bar{a})$

similarly $|Z-b|^2=|Z|^2+|b|^2-2\text{Re}(Z\bar{b})$

The given equation becomes $2|Z|^2 + |a|^2 + |b|^2 - 2\text{Re}(Z(\bar{a} + \bar{b})) = k$

$$|Z|^2 - 2\text{Re}\left[\frac{Z(\bar{a}+\bar{b})}{2}\right] + \frac{|a+b|^2}{4} = \frac{k}{2} + \frac{1}{4}|a+b|^2 - \frac{|a|^2}{2} - \frac{|b|^2}{2}$$

i.e. $$\left|Z - \frac{a+b}{2}\right|^2 = \frac{1}{2}\left\{k - \frac{1}{2}\left[|a|^2 + |b|^2 - 2\text{Re}\, ab\right]\right\}$$

i.e. $$\left|Z - \frac{a+b}{2}\right|^2 = \frac{1}{2}\left\{k - \frac{1}{2}|a-b|^2\right\}$$

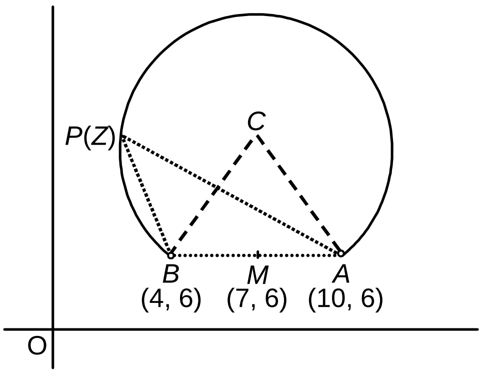
This will represent a circle with centre at $\frac{a+b}{2}$ and radius $\frac{1}{2}\sqrt{2k - |a-b|^2} \geq \frac{|a-b|^2}{2}$

Illustration – 28

Question: **Let $Z_1 = 10 + 6i; Z_2 = 4 + 6i$. If Z be any complex number such that arg $\left(\frac{Z - Z_1}{Z - Z_2}\right) = \pi/4$. prove that the locus of Z is $|Z - 7 - 9i| = 3\sqrt{2}$.**

Solution: $\because$ $\arg\left(\frac{Z - Z_1}{Z - Z_2}\right) = \arg(Z - Z_1) - \arg(Z - Z_2) = \frac{\pi}{4}$

$\Rightarrow$ the join of $A(Z_1)$ and $B(Z_2)$ subtends at $P(Z)$ an angle = $\frac{\pi}{4}$



Hence the locus of Z is a segment of a circle drawn on AB to contain the angle $\frac{\pi}{4}$

The point M – the midpoint of AB – is (7, 6) and the centre is $C(7, 9)$.
The locus of Z is the circle (segment) drawn to contain angle at $\pi/4$. The radius of the circle is $= \sqrt{9+9} = 3\sqrt{2}$

It is therefore true that Z lies on $|Z - (7 + 9i)| = 3\sqrt{2}$

But it is not true that every point Z on $|Z - (7+9i)| = 3\sqrt{2}$ satisfies that condition $\arg\left(\frac{Z - Z_1}{Z - Z_2}\right) = \pi/4$.

Therefore the locus of Z subject to the condition $\arg\left(\frac{Z - Z_1}{Z - Z_2}\right) = \pi/4$ can only be the major segment drawn on AB.
The part of the (minor) segment lying below AB may be found to satisfy the condition $\arg\left(\frac{Z - Z_1}{Z - Z_2}\right) = -\left(\pi - \frac{\pi}{4}\right) = \frac{-3\pi}{4}$

13.3 CONIC SECTION

Parabola

Equation of parabola with focus at z_0 and directrix as $\bar{a}z + a\bar{z} + b = 0$ is given by

$$|z - z_0| = \frac{|a\bar{z} + \bar{a}z + b|}{2|a|}$$

Ellipse

Equation of ellipse with foci at z_1 and z_2 and length of major axes as $2a$ is

$$|z - z_1| + |z - z_2| = 2a$$

where $2a > |z_1 - z_2|$

Hyperbola

Equation of hyperbola with foci at z_1 and z_2 and length of transverse axes as $2a$ is

$$\big||z - z_1| - |z - z_2|\big| = 2a$$

where $2a < |z_1 - z_2|$

Illustration - 29

Question: **If $||z + 2| - |z - 2|| = a^2$, $z \in C$ representing a hyperbola for $a \in R$, then find the values of a.**

Solution: Here foci are at –2 and 2 at a distance at 4. Hence the given equation represents a hyperbola if $a^2 < 4$ i.e. $a \in (-2, 2)$.

Illustration - 30

Question: **Locate the points representing the complex numbers Z in the Argand diagram for which**

(a) $|i - 1 - 2Z| > 9$ **(b) $4 \le |2Z + i| \le 6$**

(c) $|Z + i| = |Z - 1|$ **(d) $|Z - 1|^2 + |Z + 1|^2 = 4$**

Solution: **(a)** $i - 1 - 2Z = -2\left(Z + \frac{1}{2} - \frac{i}{2}\right)$

$$|i - 1 - 2Z| = \left|-2\left[Z - \left(\frac{-1+i}{2}\right)\right]\right|$$

$$= 2\left|Z - \left(\frac{-1+i}{2}\right)\right|$$

$\therefore$ The given condition becomes $\left|Z - \left(\frac{-1+i}{2}\right)\right| > \frac{9}{2}$

This represents all points represented by Z and lying outside the circle with centre $\frac{-1+i}{2}$ $\left(i.e. \left(-\frac{1}{2}, \frac{1}{2}\right)\right)$ and radius 9/2.

(b) $2Z + i = 2\left(Z + \frac{i}{2}\right)$

$$\therefore |2Z + i| = 2\left|Z + \frac{1}{2}i\right|$$

$\therefore 4 \le |2Z + i| \le 6$ gives

$$4 \le 2\left|Z + \frac{1}{2}i\right| \le 6 \; i.e. \; 2 \le \left|Z + \frac{i}{2}\right| \le 3$$

This represents the locations of all points Z on or outside the circle with centre $-\frac{i}{2}\left(i.e.\left(0,-\frac{1}{2}\right)\right)$ and radius 2; and on or inside the circle with centre at $-\frac{1}{2}i\left(i.e.\left(0,-\frac{1}{2}\right)\right)$ and radius 3. Thus it denotes the circular strip lying between two concentric circles.

(c) $|Z+i|=|Z-1|$

$|Z+i|=|Z-(-i)|$ denotes the distance of Z from $-i$ i.e. (0, –1); and $|Z-1|$ denotes the distance of Z from 1 i.e. (1, 0). Therefore $|Z+i| = |Z-1|$ is satisfied for all Z equidistant from (0, –1) and (1, 0), and thus it is perpendicular bisector of the join of (0, –1) and (1, 0), whose Cartesian equation is $x + y = 0$.

(d) $|Z-1|^2+|Z+1|^2=4$

$$|Z-1|^2+|Z+1|^2=(Z-1)(\overline{Z}-1)+(Z+1)(\overline{Z}+1)\ \ (\because |Z|^2=Z\overline{Z})$$
$$=Z\overline{Z}-(Z+\overline{Z})+1+Z\overline{Z}+(Z+\overline{Z})+1$$
$$=2Z\overline{Z}+2$$

The requirement is $2Z\overline{Z}+2=4$ $i.e.$ $Z\overline{Z}=1$ i.e. $|Z|^2=1$ $i.e.$ $|Z|=1$. Thus the locus of Z subject to the given condition is the unit circle $|Z| = 1$.

PROFICIENCY TEST–II

The following questions deal with the basic concepts of this section. Answer the following briefly. Go to the next section only if your score is at least 80%. Do not consult the Study Material while attempting the questions.

1. Solve $x^7 + 1 = 0$

2. Find all non-zero complex number satisfying $|z| + z^2 = 0$.

3. If 1, ω, ω^2 are cube roots of unity prove that

$(1 - \omega + \omega^2)(1 - \omega^2 + \omega^4)(1 - \omega^4 + \omega^8)$ ……… upto $2n$ factors $= 2^{2n}$.

4. If 1, α, α^2 ……….. α^{n-1} are nth roots of unity then find the value of

$(3 - \alpha)(3 - \alpha^2)(3 - \alpha^3)$ ……… $(3 - \alpha^{n-1})$

5. If $\frac{z_1 - z_2}{z_3 - z_4}$ in purely real for four complex numbers then, these complex numbers are collinear. (True/False)

6. The quadratic equation $|z|^2 + z|z| + z^2 = 0$ represents pair of rays. (True/False)

7. $|z - i| + |z + i| = 2$ is the equation of an ellipse. (True/False)

8. $1 < |z - 2 - 3i| < 4$ represents circular strip between two concentric circles with centre $(2 + 3i)$ and radii 1 and 4. (True/False)

ANSWERS TO PROFICIENCY TEST–II

1. $x = e^{\frac{i(2k+1)\pi}{7}}$ k = 0, 1, 2, …….. 6.

2. $z = i$

4. $\frac{3^n - 1}{2}$

5. True

6. True

7. False

8. True

SOLVED OBJECTIVE EXAMPLES

***Example* 1:**

If z_1 and z_2 be two complex numbers such that $|z_1 + z_2| = |z_1 - z_2|$ then a value of amp z_1 – amp z_2 is equal to

(a) $\frac{\pi}{2}$ **(b)** $\frac{3\pi}{2}$ **(c)** π **(d) none of these**

Solution:

Let $z_1 = a + bi$ and $z_2 = c + di$ where a, b, c, d are reals.

Now $|z_1 + z_2| = |z_1 - z_2|$

$\Rightarrow \quad |a + c + i(b + d)| = |(a - c) + i(b - d)|$

$\Rightarrow \quad (a + c)^2 + (b + d)^2 = (a - c)^2 + (b - d)^2$

$\Rightarrow \quad 4ac + 4bd = 0$

$\Rightarrow \quad ac + bd = 0$

If A and B be the points in XOY plane representing the complex numbers z_1 and z_2 respectively, then $A \equiv (a, b)$ and $B \equiv (c, d)$

$\therefore \quad \text{amp } z_1 - \text{amp } z_2 = \angle XOA - \angle XOB = \angle AOB = \frac{\pi}{2}$

($\because$ slope of $OA = \frac{b}{a}$ and slope of $OB = \frac{d}{c}$ $\Rightarrow$ (slope of OA) (slope of OB) $= \frac{b}{a}\frac{d}{c} = -1$ as $ac + bd = 0$)

Hence answer is (a)

***Example* 2:**

If $z^2 + z + 1 = 0$ then the value of $\left(z + \frac{1}{z}\right)^2 + \left(z^2 + \frac{1}{z^2}\right)^2 + \left(z^3 + \frac{1}{z^3}\right)^2 + ... + \left(z^{21} + \frac{1}{z^{21}}\right)^2$ is equal to

(a) 21 **(b) 42** **(c) 0** **(d) none of these**

Solution:

$z^2 + z + 1 = 0 \quad \Rightarrow \quad z = \omega$ or ω^2

If $z = \omega$, then $\frac{1}{z} = \omega^2$ and if $z = \omega^2$, then $\frac{1}{z} = \omega$. So, we may take $z = \omega$.

When n is a multiple of 3, then $\left(z^n + \frac{1}{z^n}\right)^2 = \left(\omega^n + \frac{1}{\omega^n}\right)^2 = (1+1)^2 = 4$

and when n is not a multiple of 3, then

$$\left(z^n + \frac{1}{z^n}\right)^2 = \left(\omega^n + \frac{1}{\omega^n}\right)^2 = \left(\omega^n + \frac{\omega^{3n}}{\omega^n}\right)^2$$

$= \quad (\omega^n + \omega^{2n})^2 = (-1)^2 = 1$ ($\because$ when n is not a multiple of 3, then $\omega^n + \omega^{2n} = -1$)

This means that in the given series 7 brackets have value 4 each and the remaining 14 brackets have value 1 each. So, the sum of the series $= 7 \times 4 + 14 \times 1 = 42$.

Hence answer is (b)

Example 3:

If ω is a non-real cube root of unity then $(a + b\,\omega + c\,\omega^2)^3 + (a + b\,\omega^2 + c\,\omega)^3$ is equal to

(a) $(a + b - c)(b + c - a)(c + a - b)$ **(b) $(a - b - c)(b - c - a)(c - a - b)$**

(c) $(2a - b - c)(2b - c - a)(2c - a - b)$ **(d) none of these**

Solution:

We know that $A^3 + B^3 = (A + B)(A\omega + B\omega^2)(A\omega^2 + B\omega)$

Substituting $A = a + b\omega + c\omega^2$ and $B = a + b\omega^2 + c\omega$, we obtain

$$(a + b\omega + c\omega)^3 + (a + b\omega^2 + c\omega)^3$$

$$= (a + b\omega + c\omega^2 + a + b\omega^2 + c\omega) \times (a\omega + b\omega^2 + c\omega^3 + a\omega^2 + b\omega^4 + c\omega^3)$$
$$\times (a\omega^2 + b\omega^3 + c\omega^4 + a\omega^4 + b\omega^3 + c\omega^2)$$

$$= (2a - b - c)(2c - a - b)(2b - a - c) \qquad (\because \omega^4 = \omega^3\omega = \omega \text{ and } \omega^2 + \omega = -1)$$

Hence answer is (c)

Example 4:

If $z_1 = a + bi$ and $z_2 = c + di$; $a, b, c, d \in R$, be two complex numbers such that $|z_1| = |z_2| = 1$ and $\text{Re}(z_1\,\bar{z}_2) = 0$ then for $\alpha_1 = a + ic$ and $\alpha_2 = b + id$,

(a) $|\alpha_1| \neq 1$ **(b) $|\alpha_2| \neq 1$** **(c) $\text{Re}\,(\alpha_1\,\bar{\alpha}_2) = 0$** **(d) $\text{Re}(\alpha_1\,\bar{\alpha}_2) = 1$**

Solution:

$|z_1| = 1, |z_2| = 1$

$\Rightarrow a^2 + b^2 = 1$ and $c^2 + d^2 = 1$ …(i)

Also $\text{Re}\,(z_1\bar{z}_2) = 0$

$\Rightarrow \text{Re}\,\{(a + ib)(c - id)\} = 0$

$\Rightarrow ac + bd = 0$ …(ii)

$\Rightarrow ac = -bd \quad \Rightarrow \quad a^2c^2 = b^2d^2$

$\Rightarrow a^2c^2 = (1 - a^2)(1 - c^2)$ (using (i))

$\Rightarrow a^2 + c^2 = 1 \quad \Rightarrow \quad |\alpha_1| = 1$

So, $a^2 + b^2 = a^2 + c^2 = c^2 + d^2 = 1$

$\Rightarrow b^2 = c^2$ and $a^2 = d^2$

$\Rightarrow b^2 + d^2 = a^2 + c^2 = 1$

$\Rightarrow |\alpha_2| = 1$

Now $\text{Re}\,(\alpha_1\bar{\alpha}_2)$ = Respectively $\{(a + ic)(b - id)\} = ab + cd$

and $(ab + cd)^2 = a^2b^2 + c^2d^2 + 2abcd + a^2c^2 + b^2d^2 + 2abcd \quad (\because c^2 = b^2)$

$= (ac + bd)^2 = 0$ (using (ii))

$\Rightarrow ab + cd = 0$

$\Rightarrow \text{Re}\,(\alpha_1\bar{\alpha}_2) = 0$

Hence answer is (c)

Example 5:

The sum of the series

$2(\omega + 1)(\omega^2 + 1) + 3(2\omega + 1)(2\omega^2 + 1) + 4(3\omega + 1)(3\omega^2 + 1) + \ldots. + (n + 1)(n\omega + 1)(n\omega^2 + 1) =$

(a) $\left(\dfrac{nn(n+1)}{2}\right)^2$ **(b) $\left(\dfrac{n(n+1)}{2}\right)^2 - n$** **(c) $\left(\dfrac{n(n+1)}{2}\right)^2 + n$** **(d) none of these**

Solution:

Sum of the given series

$$= \sum_{r=1}^{n} (r+1)(r\omega+1)(r\omega^2+1)$$

$$= \sum_{r=1}^{n} (r+1)(r^2 + r(\omega+\omega^2)+1)$$

$= \sum_{r=1}^{n} (r+1)(r^2 - r + 1) = \sum_{r=1}^{n} (r^3 + 1)$

$= \left(\frac{n(n+1)}{2}\right)^2 + n$

Hence answer is (c)

***Example* 6:**

If 1, $\alpha_1, \alpha_2, \alpha_3, \ldots\ldots, \alpha_{n-1}$ are the n, nth roots of unity then $(1+\alpha_1)(1+\alpha_2)(1+\alpha_3)\ldots\ldots(1+\alpha_{n-1})$ is equal to

(a) 1 **(b)** $\frac{1+(-1)^n}{2}$ **(c)** $\frac{1-(-1)^n}{2}$ **(d)** -1

Solution:

As 1, $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ are n, nth roots of unity, therefore,

$x^n - 1 = (x-1)(x-\alpha_1)(x-\alpha_2)\ldots\ldots(x-\alpha_{n-1})$

$\Rightarrow (x-\alpha_1)(x-\alpha_2)\ldots\ldots(x-\alpha_{n-1}) = \frac{x^n - 1}{x-1} = x^{n-1} + x^{n-2} + \ldots\ldots + x^2 + x + 1$

Put $x = -1$, to obtain

$(-1-\alpha_1)(-1-\alpha_2)(-1-\alpha_3)\ldots\ldots(-1-\alpha_{n-1})$

$= (-1)^{n-1}(1+\alpha_1)(1+\alpha_2)\ldots..(1+\alpha_{n-1}) = \frac{(-1)^{n-1}(1-(-1)^n)}{1-(-1)}$

$\Rightarrow (1+\alpha_1)(1+\alpha_2)\ldots..(1+\alpha_{n-1}) = \frac{1-(-1)^n}{2}$

Hence answer is (c)

***Example* 7:**

If $z = x + yi$, $w = \frac{1-iz}{z-i}$ and $|w| = 1$, then

(a) ***z*** **lines on the imaginary axis** **(b)** ***z*** **lies on the real axis**

(c) ***z*** **lies on the unit circle** **(d)** **none of these**

Solution:

$|\omega| = 1 \Leftrightarrow \left|\frac{1-iz}{z-i}\right| = 1$

$\Leftrightarrow \quad |1 - i(x+y)| = |x + yi - i| \quad \Leftrightarrow \quad (1+y)^2 + x^2 = x^2 + (y-1)^2$

$\Leftrightarrow \quad 4y = 0$

$\Leftrightarrow \quad y = 0$

$\Leftrightarrow$ z lies on the real axis.

Hence answer is (b)

***Example* 8:**

If $a = \text{cis}\,\alpha$, $b = \text{cis}\beta$, $c = \text{cis}\gamma$ and $\frac{a}{b} + \frac{b}{c} + \frac{c}{a} = 1$, then $\cos(\alpha-\beta) + \cos(\beta-\gamma) + \cos(\gamma-\alpha) =$

(a) 3/2 **(b)** –3/2 **(c)** 0 **(d)** 1

Solution:

$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} = 1$

$\Rightarrow \quad \frac{\text{cis}\,\alpha}{\text{cis}\beta} + \frac{\text{cis}\,\beta}{\text{cis}\gamma} + \frac{\text{cis}\,\gamma}{\text{cis}\alpha} = 1 \quad \Rightarrow \text{cis}(\alpha-\beta) + \text{cis}(\beta-\gamma) + \text{cis}(\gamma-\alpha) = 1$

Equating real parts of both sides

$\Rightarrow \quad \cos(\alpha-\beta) + \cos(\beta-\gamma) + \cos(\gamma-\alpha) = 1$

Hence answer is (d)

***Example* 9:**

If $|z - 2 - 2i| = 1$ then the minimum value of $|z|$ is

(a) $2\sqrt{2} - 1$ (b) $2\sqrt{2}$ (c) $2\sqrt{2} + 1$ (d) $2\sqrt{2} - 2$

Solution:

Given $|\mathbf{z} - 2 - 2i| = 1$

Now $|2 + 2i| = |z - (z - 2 - 2i)|$

$\leq |z| + |z - 2 - 2i|$

$\Rightarrow \sqrt{2^2 + 2^2} \leq |z| + 1 \Rightarrow |z| \geq 2\sqrt{2} - 1$

Hence minimum value of $|z|$ is $2\sqrt{2} - 1$.

Note that $z = \left(2 - \frac{1}{\sqrt{2}}\right) + \left(2 - \frac{1}{\sqrt{2}}\right) i$ satisfies $|z - (2 + 2i)| = 1$

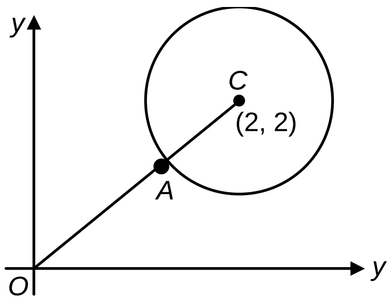
and $\left|\left(2 - \frac{1}{\sqrt{2}}\right) + \left(2 - \frac{1}{\sqrt{2}}\right) i\right| (2\sqrt{2} - 1)$.

Alternatively,

$|z - (2 + 2i)| = 1$ is a circle with centre at $C(2, 2)$ and radius 1. Let OC meet the circle at A then minimum $|z| = |OA|$

$= |OC| - |AC|$

$= \sqrt{2^2 + 2^2} - 1 = 2\sqrt{2} - 1$.



The point A can be found by solving $|z - (2 + 2i)| = 1$ and the equation of the line OC i.e. $(x - 2)^2 = 1$ and $y = x$.

Hence answer is (a)

***Example* 10:**

The point represented by the complex number $2 - i$ is rotated about origin through an angle $\pi/2$ in the clockwise direction. The complex number corresponding to new position of the point is

(a) $1 + 2i$ (b) $-1 - 2i$ (c) $2 + i$ (d) $-1 + 2i$

Solution:

Let Amp $(2 - i) = \theta$, then $2 - i = \sqrt{2^2 + 1^2}$ cis $\theta = \sqrt{5}$ cisθ

$\Rightarrow \sqrt{5}\cos\theta = 2$ and $\sqrt{5}\sin\theta = -1$.

New number = $\sqrt{5}$ cis $\left(\theta - \frac{\pi}{2}\right)$ ($\because$ rotation is clockwise, amplitude is reduced by $\pi/2$)

$= \sqrt{5}\left(\cos\left(\theta - \frac{\pi}{2}\right) + i\sin\left(\theta - \frac{\pi}{2}\right)\right)$

$= \sqrt{5}(\sin\theta - i\cos\theta)$

$= \sqrt{5}\sin\theta - i\sqrt{5}\cos\theta = -1 - 2i$

Alternatively, the number is divided by i ; so the new number = $\frac{2 - i}{i} = -1 + \frac{2}{i}$ = $-1 - 2$i.

Hence answer is (b)

***Example* 11:**

Let $\omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$, then the value of the determinant $\begin{vmatrix} 1 & 1 & 1 \\ 1 & -1-\omega^2 & \omega^2 \\ 1 & \omega^2 & \omega^4 \end{vmatrix}$ is

(a) 3ω **(b) $3\omega(\omega - 1)$** **(c) $3\omega^2$** **(d) $3\omega(1 - \omega)$**

Solution:

Given determinant

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & -1-\omega^2 & \omega^2 \\ 1 & \omega^2 & \omega^4 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega^4 \end{vmatrix} = \begin{vmatrix} 3 & 1 & 1 \\ 0 & \omega & \omega^2 \\ 0 & \omega^2 & \omega^4 \end{vmatrix}$$

operator $C_1 \to C_1 + C_2 + C_3$.

Hence answer is (b)

***Example* 12:**

For all complex numbers z_1, z_2 satisfying $|z_1| = 12$ and $|z_2 - 3 - 4i| = 5$, the minimum value of $|z_1 - z_2|$ is

(a) 0 **(b) 2** **(c) 7** **(d) 17**

Solution:

First, we note that

$$|z_2| = |(z_2 - 3 - 4i) + (3 + 4i)| \le |z_2 - 3 - 4i| + |3 + 4i| = 5 + 5 = 10$$

Hence $|z_1 - z_2| \ge \big||z_1| - |z_2|\big|$

$= |12 - |z_2|| \ge |12 - 10| = 2$ $(\because |z_2| \le 10 \;\therefore\; -|z_2| \ge -10)$

Hence answer is (b)

***Example* 13:**

Let z_1 and z_2 be nth roots of unity which subtend a right angle at the origin, then n must be of the form

(a) $4k$ **(b) $4k + 3$** **(c) $4k + 2$** **(d) $4k + 11$**

Solution:

The n, nth roots of unity are given by

$$\text{cis}\left(\frac{2k\pi}{n}\right), k = 0, 1, 2, \ldots\ldots n-1$$

Let $z_1 = \text{cis}\left(\frac{2k_1\pi}{n}\right)$

and $z_2 = \text{cis}\left(\frac{2k_2\pi}{n}\right)$

where k_1 and k_2 are distinct integers from the set {0, 1, 2, 3, …., $n - 1$}.

As z_1 and z_2 subtend a right angle at the origin, therefore,

$$\left|\frac{2k_1\pi}{n} - \frac{2k_2\pi}{n}\right| = \frac{\pi}{2}, k_1 \ne k_2$$

$\Rightarrow \frac{2\pi}{n}|k_1 - k_2| = \frac{\pi}{2} \quad \Rightarrow \quad 4|k_1 - k_2| = n$

So, n = 4(a positive integer).

Hence answer is (a)

Example **14:**

z and ω are two non-zero complex numbers such that $|z| = |\omega|$ and Arg z + Arg $\omega = \pi$, then z =

(a) $\overline{\omega}$ **(b)** $-\overline{\omega}$ **(c)** ω **(d)** $-\omega$

Solution:

Let $|z| = |\omega| = r$ and Arg $\omega = \theta$

then $\omega = r$ cis θ and Arg $z = \pi - \theta$

Hence z = r cis $(\pi - \theta)$

$= r\{\cos(\pi - \theta) + i\sin(\pi - \theta)\}$

$= r(-\cos\theta + i\sin\theta) = -r(\cos\theta - i\sin\theta) = -\overline{\omega}$.

Hence answer is (b)

Example **15:**

The locus of the centre of a circle which touches the circles $|z - z_1| = a$ and $|z - z_2| = b$ externally (z, z_1 and z_2 are complex numbers) will be

(a) an ellipse **(b) a hyperbola** **(c) a circle** **(d) none of these**

Solution:

Let $A(z_1)$, $B(z_2)$ be the centres of given circles and P be the centre of the variable circle which touches given circles externally, then

$|AP| = a + r$ and $|BP| = b + r$, where r is the radius of the variable circle. On subtraction, we get

$|AP| - |BP| = a - b$

$\Rightarrow$ $||AP| - |BP|| = |a - b|$, a constant. Hence locus of P is

(i) right bisector of $[AB]$ if $a = b$.

(ii) a hyperbola if $|a - b| < |AB| = |z_2 - z_1|$

(iii) an empty set if $|a - b| > |AB| = |z_2 - z_1|$

(iv) Set of all points on line AB except those which lie between A and B if $|a - b| = |AB| \neq 0$.

Hence answer is (d)

SOLVED SUBJECTIVE EXAMPLES

***Example* 1:**

Find the value of $(x^2+5x)^2 + x(x+5)$ for $x = \dfrac{-5+i\sqrt{3}}{2}$

Solution:

$$x+5 = \frac{-5+i\sqrt{3}}{2} + 5 = \frac{5+i\sqrt{3}}{2}$$

$$\therefore x(x+5) = \left(\frac{-5+i\sqrt{3}}{2}\right)\left(\frac{5+i\sqrt{3}}{2}\right) = \frac{(-5)\,5+3i^2}{4}$$

$$= \frac{-25-3}{4} = -7$$

$\therefore$ The (required) value = $(-7)^2 - 7 = 49 - 7 = 42$

***Example* 2:**

Find two complex numbers satisfying the conditions that

(i) the sum of their real parts is 3

(ii) the product of their real parts is 2

(iii) their product is 5 – *i*

Solution:

Take the complex numbers as $a + ib$, $p + iq$

so that $a+p=3;\ ap=2 \Rightarrow \left.\begin{matrix} a=2 \\ p=1 \end{matrix}\right\}$ or $\left.\begin{matrix} a=1 \\ p=2 \end{matrix}\right\}$

Also $(a + ib)(p + iq) = ap - bq + i(bp + aq) = 5 - i$

Given $ap - bq = 5$; $aq + bp = -1$

Taking $a = 2$, $p = 1$; $bq = -3$ and $b + 2q = -1$

This gives $\left.\begin{matrix} b=-3 \\ q=1 \end{matrix}\right\}$ or $\left.\begin{matrix} 2+2i \\ 1-\frac{3}{2}i \end{matrix}\right\}$

The numbers are $\left.\begin{matrix} 2-3i \\ 1+i \end{matrix}\right\}$ or $\left.\begin{matrix} 2+2i \\ 1-\frac{3}{2}i \end{matrix}\right\}$

Thus there are two pairs of a complex numbers satisfying the requirements.

It may be verified that $a = 1$, $p = 2$, give the same set of numbers.

***Example* 3:**

Prove that

(i) $|Z_1+Z_2|^2 + |Z_1-Z_2|^2 = 2(|Z_1|^2+|Z_2|^2)$

(ii) using above result, prove that $\left|\alpha-\sqrt{a^2-\beta^2}\right| + \left|\alpha+\sqrt{\alpha^2+\beta^2}\right| = |\alpha+\beta| + |\alpha-\beta|$**,**

where α, β are complex numbers.

Solution:

$$|Z_1+Z_2|^2 = (Z_1+Z_2)(\bar{Z}_1+\bar{Z}_2) = Z_1\bar{Z}_1 + Z_2\bar{Z}_2 + Z_1\bar{Z}_2 + Z_2\bar{Z}_1$$

$$|Z_1-Z_2|^2 = (Z_1-Z_2)(\bar{Z}_1-\bar{Z}_2) = Z_1\bar{Z}_1 + Z_2\bar{Z}_2 - Z_1\bar{Z}_2 - Z_2\bar{Z}_1$$

Adding, $|Z_1+Z_2|^2 + |Z_1-Z_2|^2 = 2(Z_1\bar{Z}_1 + Z_2\bar{Z}_2)$

$$= 2\,(|Z_1|^2 + |Z_2|^2)$$

Now, for the second part,

$$\left|\alpha-\sqrt{\alpha^2-\beta^2}\right|+\left|\alpha+\sqrt{\alpha^2-\beta^2}\right|$$

$$=\frac{1}{2}\left\{\left|2\alpha-2\sqrt{\alpha^2-\beta^2}\right|+\left|2\alpha+2\sqrt{\alpha^2-\beta^2}\right|\right\}$$

$$=\frac{1}{2}\left\{\left|\alpha+\beta+\alpha-\beta-2\sqrt{\alpha^2-\beta^2}\right|+\left|\alpha+\beta+\alpha-\beta+2\sqrt{\alpha^2-\beta^2}\right|\right\}$$

$$=\frac{1}{2}\left\{\left|\sqrt{\alpha+\beta}-\sqrt{\alpha-\beta}\right|^2+\left|\sqrt{\alpha+\beta}+\sqrt{\alpha-\beta}\right|^2\right\}$$

$$=\frac{1}{2}\left\{|Z_1-Z_2|^2+|Z_1+Z_2|^2\right\}$$

$$=\frac{1}{2}\left\{2(|Z_1|^2+|Z_2|^2)\right\}$$

$$=\left|\sqrt{\alpha+\beta}\right|^2+\left|\sqrt{\alpha-\beta}\right|^2$$

$$=|\alpha+\beta|+|\alpha-\beta|$$

Example 4:

If Z be a complex number with $|Z| = 1$, imaginary part of $Z \neq 0$, show that Z can be represented as $\frac{c+i}{c-i}$ where c is real.

Solution:

Since $|Z| = 1$, Z can be represented as $(\cos\theta + i\sin\theta)$

$\therefore \quad Z = (\cos\theta + i\sin\theta)$

$$=\left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)\left(\cos\frac{\theta}{2}+i\frac{\sin\theta}{2}\right)$$

$$=\frac{\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}-i\sin\frac{\theta}{2}}$$

$$=\frac{\cot\frac{\theta}{2}+i}{\cot\frac{\theta}{2}-i}\left(\text{Dividing by }\sin\frac{\theta}{2}\right)$$

$$=\frac{c+i}{c-i}\text{ where c = }\cot\frac{\theta}{2}\text{ is real.}$$

Example 5:

For every real $b \geq 0$, find all the complex numbers Z satisfying $2|Z| - 4bZ + 1 + ib = 0$.

Solution:

Let $Z = x + iy$. The equation is

$$2\sqrt{x^2+y^2}-4b(x+iy)+1+ib=0$$

Real part: $2\sqrt{x^2+y^2}-4bx+1=0$ … (i)

Imaginary Part: $-4by+b=0$ …(ii)

From (ii) either $b = 0$ and in that case from (i), $2\sqrt{x^2+y^2}+1=0$ and this equation is not satisfied for any (x, y)

$\therefore$ $b = 0$, there is no solution for the equation, If $b \neq 0$ but > 0; $-4y + 1 = 0$ from (ii)

i.e. $y = \frac{1}{4}$

Substituting $y = \frac{1}{4}$ in (i)

$$2\sqrt{x^2 + \frac{1}{16}} = 4bx - 1 \quad \text{... (iii)}$$

This requires that $4bx - 1 > 0$ i.e. $x > \frac{1}{4b}$ and $b > 0$ and hence $x > 0$

Squaring (iii)

$$4\left(x^2 + \frac{1}{16}\right) = 16b^2x^2 - 8bx + 1$$

$$x^2(16b^2 - 4) - 8bx + \frac{3}{4} = 0$$

Roots are $\frac{8b \pm \sqrt{16b^2 + 12}}{2(16^2 - 4)} = \frac{4b \pm \sqrt{4b^2 + 3}}{16b^2 - 4}$

If $16b^2 - 4 < 0$, which in effect means that $b < \frac{1}{2}$ (note already $b > 0$), the values of x are such that

(i) for the + sign $x < 0$ while the requirement is $x > 0$

(ii) for the – sign, even if $x > 0$, the condition $x > \frac{1}{4b}$ is not satisfied.

$\therefore$ For $0 < b < \frac{1}{2}$, there is no solution.

For $b > \frac{1}{2}$, the solution is

$$Z = \frac{4b + \sqrt{4b^2 + 3}}{16b^2 - 4} + \frac{i}{4}$$

***Example* 6:**

For complex numbers z and w, prove that $|z|^2 w - |w|^2 z = z - w$ if and only if $z = w$ or $z\overline{w} = 1$.

Solution:

$$\frac{z}{w} = \frac{|z|^2 + 1}{|w|^2 + 1} = \text{purely real number}$$

$\Rightarrow \frac{z}{w}$ is purely real i.e., $\frac{z}{w} = \overline{\left(\frac{z}{w}\right)} \Rightarrow z\overline{w} = \overline{z}w$ ...(i)

$$|z|^2 w - |w|^2 z = z - w$$

$$z\overline{z}w - w\overline{w}z = z - w$$

from (i), $z\overline{w}(z - w) = z - w$ ...(ii)

$(z\overline{w} - 1)(z - w) = 0 \Rightarrow z = w$ or $z\overline{w} = 1$

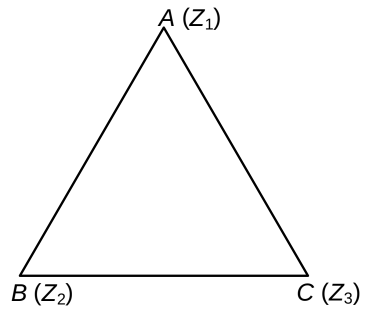
Conversely if $z = w$, then LHS = RHS = 0

$z\overline{w} = 1 \Rightarrow$ then from (i) and (ii) L.H.S. = R.H.S. = $z - w$

***Example* 7:**

Show that the triangle whose vertices are the points represented by the complex numbers Z_1, Z_2 and Z_3 on the argand diagram is equilateral if and only if $\frac{1}{Z_2 - Z_3} + \frac{1}{Z_3 - Z_1} + \frac{1}{Z_1 - Z_2} = 0$ (OR) equivalently $Z_1^2 + Z_2^2 + Z_3^2 = Z_1Z_2 + Z_2Z_3 + Z_3Z_1$.

Solution:



ABC is the equilateral triangle formed of $A(Z_1)$; $B(Z_2)$ *and* $C(Z_3)$

$$Z_3 - Z_1 = (Z_2 - Z_1)\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$

(*AC* is obtained from *AB* by a rotation anticlockwise through an angle π/3) Lengthwise *AC* = *AB*

$$\frac{1}{Z_3 - Z_1} = \frac{1}{Z_1 - Z_2}\left\{-\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right\} \quad \text{...(i)}$$

Similarly

$$\frac{1}{Z_2 - Z_3} = \frac{1}{Z_1 - Z_2}\left\{-\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right\} \quad \text{... (ii)}$$

Adding (i) and (ii)

$$\frac{1}{Z_3 - Z_1} + \frac{1}{Z_2 - Z_3} = -\frac{1}{Z_1 - Z_2}$$

i.e. $$\frac{1}{Z_3 - Z_1} + \frac{1}{Z_2 - Z_3} + \frac{1}{Z_1 - Z_2} = 0 \quad \text{... (iii)}$$

This may be equivalently written in the form

$$\sum (Z_1 - Z_2)(Z_3 - Z_1) = 0$$

i.e. $$\sum Z_1(Z_3 - Z_1) - \sum Z_2(Z_3 - Z_1) = 0$$

i.e. $$Z_1^2 + Z_2^2 + Z_3^2 = Z_1Z_2 + Z_3Z_1 + Z_2Z_3 \qquad \left(\because \sum Z_2(Z_3 - Z_1) = 0\right) \text{...(iv)}$$

The condition for Z_1, Z_2 *and* Z_3 to form an equilateral triangle is given in one of the two equivalent forms given by (iii) and (iv).

Let us prove the converse also

Assume $$\frac{1}{Z_2 - Z_3} + \frac{1}{Z_3 - Z_1} + \frac{1}{Z_1 - Z_2} = 0 \quad \text{... (A)}$$

If $\quad p = Z_2 - Z_3, q = Z_3 - Z_1; r = Z_1 - Z_2; p + q + r = 0$

$\therefore \quad p(q + r) + rq = 0 \Rightarrow p(-p) + qr = 0 \Rightarrow p^2 = qr$

$\therefore \quad p^2 = qr; \quad \bar{p}^2 = \bar{q}\,\bar{r}$

$\therefore \quad p^2\bar{p}^2 = q\bar{q}\,r\bar{r} \Rightarrow (p\bar{p})^2\,(p\bar{p}) = (q\bar{q})\,(r\bar{r})\,(p\bar{p})$

Similarly it is possible to prove $p\bar{p}\;q\bar{q}\;r\bar{r} = (q\bar{q})^3 = (r\bar{r})^3$

This gives $p\bar{p} = q\bar{q} = r\bar{r}$

i.e. $\quad |p^2| = |q|^2 = |r|^2$

i.e. $\quad |Z_2 - Z_3| = |Z_3 - Z_1| = |Z_1 - Z_2|$

$\therefore$ The triangle is equilateral

Let us also prove the converse from the other condition

namely $Z_1^2 + Z_2^2 + Z_3^2 - Z_1Z_2 - Z_2Z_3 - Z_3Z = 0 \quad$ ...(B)

ω, ω^2 being the two imaginary cube roots of unity; (B) may be written as

$$(Z_1 + \omega Z_2 + \omega_2 Z_3)(Z_1 + \omega^2 Z_2 + \omega Z_3) = 0$$

Hence $\quad Z_1 - Z_2 = -Z_2 - \omega Z_2 - \omega^2 Z_3$

$\therefore \quad = -Z_2(1+\omega) - \omega^2 Z_3$

$= -Z_2(-\omega^2) - \omega^2 Z_3$

$Z_1 - Z_2 = \omega^2 (Z_2 - Z_3)$

$\therefore |Z_1 - Z_2| = |\omega^2||Z_2 - Z_3| \Rightarrow |Z_1 - Z_2| = |Z_2 - Z_3|$

and similarly it can be proved by combining the terms differently $|Z_1 - Z_3| = |Z_2 - Z_3|$.

Hence $|Z_1 - Z_2| = |Z_2 - Z_3| = |Z_3 - Z_1|$

$\therefore$ The triangle is equilateral.

***Example* 8:**

Find all non-zero complex numbers satisfying $\bar{Z} = iZ^2$.

Solution:

Let $Z = x + iy$; $\bar{Z} = x - iy$; $Z^2 = x^2 - y^2 + 2ixy$

$\therefore$ The equation is $x - iy = i(x^2 - y^2 + 2ixy)$

$\therefore$ Equating real and imaginary parts

$x = -2xy \quad \ldots$ (i)

$-y = x^2 - y^2 \quad \ldots$(ii)

(i) gives either $x = 0$, in that case $y = 0$; $y = 1$

or $\quad y = -\frac{1}{2}$, in that case $\frac{1}{4} + \frac{1}{2} = x^2$

$\therefore \quad x = \pm \frac{\sqrt{3}}{2}$

$\therefore$ The non-zero Z, satisfying the equation are

$$Z_1 = i \ ; \ Z_2 = \frac{\sqrt{3}}{2} - \frac{1}{2}i \ ; \ Z_3 = -\frac{\sqrt{3}}{2} - \frac{1}{2}i$$

***Example* 9:**

If *A*, *B*, *C*, *D* are four points in a plane forming a quadrilateral *ABCD*, prove that $AC \cdot BD \le AB \cdot CD + AD \cdot BC$. When does the equality exist?

Solution:

Let the four points *A*, *B*, *C* and *D* have associate complex numbers; Z_1, Z_2, Z_3 and Z_4

First factor Z_4 fixed, Z_1, Z_2, Z_3 cyclically changed. For II terms Z_2, Z_3, Z_1 cyclically changed.

we have

$(Z_1 - Z_4)(Z_2 - Z_3) + (Z_2 - Z_4)(Z_3 - Z_1) + (Z_3 - Z_4)(Z_1 - Z_2) = 0$

$\therefore -(Z_3 - Z_1)(Z_2 - Z_4) = (Z_1 - Z_4)(Z_2 - Z_3) + (Z_3 - Z_4)(Z_1 - Z_2)$

$\therefore |(Z_3 - Z_1)(Z_2 - Z_4)| = |(Z_1 - Z_4)(Z_2 - Z_3) + (Z_3 - Z_4)(Z_1 - Z_2)|$

$\le |Z_1 - Z_4||Z_2 - Z_3| + |Z_3 - Z_4||Z_1 - Z_2|$

i.e. $AC \cdot BD \le AD \cdot BC + AB \cdot CD$

When equality exists, we have

$|(Z_1 - Z_4)(Z_2 - Z_3) + (Z_3 - Z_4)(Z_1 - Z_2)|$

$= |Z_1 - Z_4||Z_2 - Z_3| + |Z_3 - Z_4||Z_1 - Z_2|$

$\Rightarrow \quad \arg\{(z_1 - z_4)(z_2 - z_3)\} = \arg\{(z_3 - z_4)(z_1 - z_2)\}$

$\Rightarrow \quad \arg\{(z_1 - z_4)(z_2 - z_3)\} - \arg\{(z_3 - z_4)(z_1 - z_2)\} = 0$

$\Rightarrow \quad \arg\left\{\frac{(z_1 - z_4)(z_2 - z_3)}{(z_3 - z_4)(z_1 - z_2)}\right\} = 0$

i.e. when *A*, *B*, *C*, *D* are concylic points.

i.e. when Z_1, Z_2, Z_3 and Z_4 represent four points which are concylic.

Example* 10*:

Solve the equation; $Z + a\,|Z+1| + i = 0$ (a is a real number $\geq$ 1).

Solution:

Taking $Z = x + iy$, the equation reduces to $x + iy + a\sqrt{x^2 + 2x + 1 + y^2} + i = 0$.

Imaginary $= 0 \Rightarrow y = -1$

Real part $= 0 \Rightarrow x + a\sqrt{x^2 + 2x + 1 + y^2} = 0$

Eliminating y, the equation in x is

$x^2 = a^2 (x^2 + 2x + 2)$

$x^2 (a^2 - 1) + 2a^2x + 2a^2 = 0$

This gives real x only if $4a^4 - 8a^2 (a^2 - 1) \geq 0$

i.e if $-a^4 + 2a^2 \geq 0$

i.e if $0 \leq a^2 \leq 2$

and $a \geq 1$, the value of a are $1 \leq a \leq \sqrt{2}$

$x = \dfrac{-a^2 \pm a\sqrt{2 - a^2}}{a^2 - 1}, < 0$ for the negative sign and for the positive sign also $x < 0$.

Hence the solutions are;

$$Z = \left\{\frac{-a^2 \pm a\sqrt{2 - a^2}}{a^2 - 1}\right\} - i \quad \text{where } 1 \leq a \leq \sqrt{2}\,.$$

MIND MAP

COMPLEX NUMBERS

Complex numbers

- $z = x + iy$, $x, y \in \boldsymbol{R}$ and $i^2 = -1$
- Real part of $z = Re(z) = x$
- Imaginary part = $Im(z) = y$
- Modulus of $z = |z| = \sqrt{x^2 + y^2}$
- Conjugate of $z = \bar{z} = x - iy$
- Argument of $z = \theta, \pi - \theta, -\pi + \theta, -\theta$

  $\left(\text{where } \theta = \tan^{-1}\left|\frac{y}{x}\right|\right)$ according as z lies in first, second, third or fourth quadrant.

Cube roots of unity

1, ω, ω^2 are roots of $x^3 - 1 = 0$ such that

$1 + \omega + \omega^2 = 0$

$\omega^3 = 1$

Geometrically they represent vertices of an equilateral triangle inscribed in $|z| = 1$

n[th] roots of unity

Roots of $x^n - 1 = 0$ are 1, $\alpha, \alpha^2 \ldots\ldots \alpha^{n-1}$

where $\alpha^k = e^{\frac{i2k\pi}{n}}$ $k = 0, 1, \ldots. (n-1)$

$1 + \alpha + \ldots. \alpha^{n-1} = 0 \quad \Rightarrow \quad \sum_{k=0}^{n-1} e^{\frac{i2k\pi}{n}} = 0$

$\Rightarrow \quad \sum_{k=0}^{n-1} \cos\frac{2k\pi}{n} = 0$ and $\sum_{k=0}^{n-1} \sin\frac{2k\pi}{n} = 0$

Geometrically they represent vertices of an n sided regular polygon inscribed in $|z| = 1$

Rotation theorem

- If $\arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right) = \theta$

  Then $\frac{z_3 - z_1}{z_2 - z_1} = \left|\frac{z_3 - z_1}{z_2 - z_1}\right| e^{i\theta}$
- Four points z_1, z_2, z_3, z_4 are concyclic if $\frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)}$ is real.

Theory of equation with complex coefficients

An n^{th} degree equation with complex coefficients $a_n, a_{n-1}, \ldots\ldots, a_0$ is given as

$a_n z^n + a_{z-1} z^{n-1} + \ldots\ldots + a_1 z + a_0 = 0$

It has n roots say $\alpha_1, \alpha_2, \ldots\ldots\alpha_n$, and

$\sum \alpha_1 = -\frac{a_{n-1}}{a_n}$, $\sum \alpha_1\alpha_2 = \frac{a_{n-2}}{a_n}$........

Properties of modulus, conjugate and argument

- $|z| = |\bar{z}|$
- $z\bar{z} = |z|^2$
- $|z_1 z_2| = |z_1||z_2|$
- $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$
- $|z_1 \pm z_2| \le |z_1| + |z_2|$
- $|z_1 \pm z_2| \ge \big||z_1| - |z_2|\big|$
- $\overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_2$
- $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$
- $z = \bar{z} \Rightarrow$ z is purely real.

  $z = -\bar{z} \Rightarrow$ z is purely imaginary.
- Arg (z_1z_2) = Arg (z_1) + Arg (z_2) + $2k\pi$, where $k = -1, 0, 1$ which is suitable

  Arg $\left(\frac{z_1}{z_2}\right)$ = Arg z_1 – Arg z_2 + $2k\pi$, where $k = -1, 0, 1$ which is suitable, Arg $\bar{z}$ = – Arg z

Logarithm of a complex number

$\log_e(z) = \log_e|z| + i \arg(z)$

Section formula

z divides join of $A(z_1)$ and $B(z_2)$ in ratio $m : n$ then

$$z = \frac{mz_2 + nz_1}{m + n}$$

Straight line

Line passing through point z_1 and z_2 is

$$\frac{z - z_1}{z_2 - z_1} = \frac{\bar{z} - \bar{z}_1}{\bar{z}_2 - \bar{z}_1}$$

Ray

arg $(z - \alpha) = \theta$ is a ray from point α and pointed in direction of join of α to z and inclined at an angle θ to positive direction of x-axis.

Perpendicular bisector

of line joining z_1 and z_2 is $|z - z_1| = |z - z_2|$

Circle

- $|z - z_0| = r$ is circle with centre z_0 and radius r.
- $|z - z_1| = k|z - z_2|$ $(k \ne 1)$ is also a circle.
- $\arg\left(\frac{z - z_1}{z - z_2}\right) = \theta$ $(\theta \ne 0, \pi)$ represents segment of a circle described on join of z_1 and z_2.

Parabola

Equation of parabola with focus at z_0 and directrix as $\bar{a}z + a\bar{z} + b = 0$ is given by

$$|z - z_0| = \frac{|a\bar{z} + \bar{a}z + b|}{2|a|}$$

Ellipse

Equation of ellipse with foci of z_1 and z_2 and length of major axes as 2a is

EXERCISE – I

CBSE PROBLEMS

1. Find the value of $\dfrac{i^{592}+i^{590}+i^{588}+i^{586}+i^{584}}{i^{582}+i^{580}+i^{578}+i^{576}+i^{574}}$.

2. If $x = a + b$, $y = a\alpha + b\beta$ and $z = a\beta + b\alpha$, where α and β are complex cube roots of unity, show that $xyz = a^3 + b^3$.

3. Prove that $\left(\dfrac{-1+i\sqrt{3}}{2}\right)^{17}+\left(\dfrac{-1-i\sqrt{3}}{2}\right)^{17}=-1.$

4. Graph the complex numbers on the Argand plane $3 + 2i$, $-4 - 5i$, -5, $1 - 2i$.

5. Express the numbers in the form $r(\cos\theta + i\sin\theta)$:

(i) $1 + i\tan\alpha$ (ii) $1 - \sin\alpha + i\cos\alpha$.

6. Find the square root of following:

(i) $7 - 24i$ (ii) $-5+12i$

(iii) $4ab-2\left(a^2-b^2\right)i$ (iv) $x^2+\dfrac{1}{x^2}+4i\left(x-\dfrac{1}{x}\right)-6$

7. Find x and y in the following equations:

(i) $(x+iy)(2-3i)=4+i$ (ii) $\dfrac{(1+i)x-2i}{3+i}+\dfrac{(2-3i)y+i}{3-i}=i$

8. Express the following in the form of $a+ib$

(i) $\dfrac{5+\sqrt{2}\,i}{1-\sqrt{2}\,i}$ (ii) $(1-i)^4$

9. For any two complex numbers z_1 and z_2, prove that $\text{Re}(z_1z_2)=\text{Re}\,z_1\,\text{Re}\,z_2-\text{Im}\,z_1\,\text{Im}\,z_2$.

10. If $(a+ib)(c+id)(e+if)(g+ih)=A+iB$, then find value of $\left(a^2+b^2\right)\left(c^2+d^2\right)\left(e^2+f^2\right)\left(g^2+h^2\right)$.

11. Find the value of $\sqrt[4]{-64}$.

12. Find the real θ such that $\dfrac{3+2i\sin\theta}{1-2i\sin\theta}$ is purely real.

13. Find the conjugate of $\dfrac{(3-2i)(2+3i)}{(1+2i)(2-i)}$.

14. Find modulus of the following complex numbers:

(i) $\dfrac{1+2i}{1-3i}$ (ii) $\dfrac{1+i}{1-i}-\dfrac{1-i}{1+i}$

15. Find argument of following complex numbers:

(i) $\dfrac{1+i}{1-i}$ (ii) $-\sqrt{3}-i$

16. Put the following complex numbers in polar form:

(i) $\frac{1+2i}{1-3i}$ (ii) $1+i$

17. If $|2z-1|=|z-2|$, then find value of $|z|$.

18. If $x=\sqrt{2}\,i-1$ find the value of $x^4+4x^3+6x^2+4x+9$.

19. Express $(1+a^2)(1+b^2)$ as sum of two squares.

20. Find the value of $\frac{(1+i)^{4n+5}}{(1-i)^{4n+3}}$.

EXERCISE – II

NEET-SINGLE CHOICE CORRECT

1. $x+iy=\left(1-i\sqrt{3}\right)^{100}$, then (*x, y*) is equal to

(a) $\left(2^{99},\ 2^{99}\sqrt{3}\right)$
(b) $\left(2^{99},\ -2^{99}\sqrt{3}\right)$
(c) $\left(-2^{99},\ 2^{99}\sqrt{3}\right)$
(d) none of these

2. The smallest positive integer for which $(1+i)^{2n}=(1-i)^{2n}$ is

(a) 4 (b) 8 (c) 2 (d) 12

3. The value of $\sum_{n=1}^{13}\left(i^n+i^{n+1}\right)$, where $i=\sqrt{-1}$ equals

(a) i (b) $i-1$ (c) $-i$ (d) 0

4. If z_1, z_2 and z_3, z_4 are two pairs of conjugate complex numbers, then $\arg\left(\frac{z_1}{z_4}\right)+\arg\left(\frac{z_2}{z_3}\right)$ equals

(a) 0 (b) $\frac{\pi}{2}$ (c) $\frac{3\pi}{2}$ (d) π

5. The number of solutions of $Z^2+3\bar{z}=0$ is

(a) 2 (b) 3 (c) 4 (d) 5

6. The locus of *z* moving in the Argrand plane such that $\arg\left(\frac{z-2}{z+2}\right)=\frac{\pi}{2}$ is

(a) a straight line
(b) a semi-circle centred at origin and radius 2
(c) a circle centred at origin and radius $\sqrt{2}$
(d) none of these

7. One of the values of i^i is

(a) $e^{-\pi/2}$ (b) $e^{\pi/2}$ (c) e^{π} (d) $e^{-\pi}$

8. For a complex number *z*, the minimum value of $|z|+|z-2|$ is

(a) 1 (b) 2 (c) 3 (d) none of these

9. Let $z=1-t+i\sqrt{\left(t^2+t+2\right)}$, where *t* is a real parameter. The locus of *z* in the Argand plane is a part of

(a) a hyperbola
(b) an ellipse
(c) a straight line
(d) none of these

10. If $|z| \geq 3$, the least value of $\left|z+\frac{1}{z}\right|$ is

(a) $\frac{8}{3}$ (b) $\frac{3}{8}$ (c) $\frac{10}{3}$ (d) none of these

11. The value of $\sum_{k=1}^{6}\left(\sin\frac{2k\pi}{7}-i\cos\frac{2k\pi}{7}\right)$ is

(a) –1 (b) 1 (c) 0 (d) i

12. If $|z_1+z_2|=|z_1|+|z_2|$, then $\arg z_1-\arg z_2$ is equal to

(a) π (b) $-\frac{\pi}{2}$ (c) $\frac{\pi}{2}$ (d) 0

13. If $|z+i|=|z-i|$, then the locus of z is

(a) imaginary axis (b) real axis
(c) region above the x-axis (d) region below the X-axis

14. If $z_1=\frac{mz_2+z_3}{m+1}$ where $m\in R^+$, then the distance of z_1 from the line joining z_2 and z_3 is

(a) 0 (b) m (c) $\frac{m}{m+1}$ (d) $\frac{1}{m+1}$

15. If $|z-4|>|z-2|$, then

(a) $\operatorname{Re} z<3$ (b) $\operatorname{Re} z<2$ (c) $\operatorname{Re} z>2$ (d) $\operatorname{Re} z>3$

16. If $z_1=\frac{-1+3i}{2}$ and $z_2=\frac{-1-3i}{2}$, then value of $z_1^3+z_2^3-3z_1z_2$ is

(a) 1 (b) –1 (c) 3 (d) –3

17. If one root of the equation $ix^2-2(1+i)x+(2-i)=0$ is $2-i$, then the other root is

(a) $2+i$ (b) $2-i$ (c) i (d) $-i$

18. For $x_1, x_2, y_1, y_2 \in R$, if $0<x_1<x_2, y_1=y_2$ and $z_1=x_1+iy_1, z_2=x_2+iy_2$ and $z_3=\frac{z_1+z_2}{2}$, then z_1, z_2 and z_3 satisfy

(a) $|z_1|<|z_3|<|z_2|$ (b) $|z_1|>|z_3|>|z_2|$
(c) $|z_1|<|z_2|<|z_3|$ (d) $|z_1|=|z_2|=|z_3|$

19. The number of values of z satisfying both $|z|=4$ and $|z|=|z-6|$ is

(a) 2 (b) 1 (c) 4 (d) 0

20. The equation $|z-1|-|z+1|=\lambda$ represents a hyperbola if

(a) $-2<\lambda<2$ (b) $\lambda>2$
(c) $0<\lambda<2$ (d) none of these

21. If $\frac{z-1}{z+1}$ is purely imaginary, then $|z|$ is

(a) equal to 1
(b) > 1
(c) < 1
(d) > 2

22. If $\left(\frac{3}{2}+i\frac{\sqrt{3}}{2}\right)^{50}=\frac{3^{25}}{2}(x+iy)$, then the ordered pair (x, y) is equal to

(a) $(1, -\sqrt{3})$
(b) $(\sqrt{3}, 1)$
(c) $(1, \sqrt{3})$
(d) $(\sqrt{3}, -1)$

23. The equation $z\bar{z}+(4-3i)z+(4+3i)\bar{z}+5=0$ represents a circle of radius

(a) $2\sqrt{5}$
(b) $\sqrt{5}$
(c) 5
(d) $\frac{5}{2}$

24. If $M=$ greatest $|z|$ and m = least $|z|$ for the complex number satisfying $|z-(3+4i)|=1$ then the ordered pair (M, m) is

(a) (5, 4)
(b) (6, 4)
(c) (6, 5)
(d) (4, 3)

25. Locus of z, if $\left|\frac{z-1}{z+1}\right|=2$, is

(a) circle
(b) pair of straight line
(c) ellipse
(d) none of these

EXERCISE – III

IIT-JEE – SINGLE CHOICE CORRECT

1. If $x_r = \cos\frac{\pi}{2^r} + i\sin\frac{\pi}{2^r}$ then $x_1x_2x_3.....\infty$ is equal to

(a) 1
(b) –1
(c) 0
(d) none of these

2. If $\sin\alpha + \sin\beta + \sin\gamma = 0 = \cos\alpha + \cos\beta + \cos\gamma$ then $\cos 2\alpha + \cos 2\beta + \cos 2\gamma$ is equal to

(a) 0
(b) 1
(c) –1
(d) can't be determined

3. If $1, \omega, \omega^2, ..., \omega^{n-1}$ are n, n^{th} roots of unity then $(5-\omega)(5-\omega^2)....(5-\omega^{n-1})$ is equal to

(a) $\frac{5^n - 1}{4}$
(b) $\frac{5^n + 1}{4}$
(c) $\frac{5^n}{4}$
(d) $\frac{5^n + 2}{4}$

4. If z satisfies the equation $|z - 3i| + |z - 4| = 5$, then minimum value of $|z|$ is

(a) 5
(b) 12
(c) $\frac{5}{12}$
(d) $\frac{12}{5}$

5. Let z and w be two complex numbers such that $|z| \le 1, |w| \le 1$ and $|z + iw| = |z - i\overline{w}| = 2$, then z equals.

(a) 1 or i
(b) i or $-i$
(c) 1 or –1
(d) i or –1

6. If $i = \sqrt{-1}$, then $4 + 5\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^{334} + \left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^{365}$ is equal to

(a) $1 - i\sqrt{3}$
(b) $-1 + i\sqrt{3}$
(c) $i\sqrt{3}$
(d) $1 + 2i\sqrt{3}$

7. The complex numbers z_1, z_2 and z_3 satisfying $\frac{z_1 - z_3}{z_2 - z_3} = \frac{1 - i\sqrt{3}}{2}$ are the vertices of a triangle which is

(a) of area zero
(b) right angled isosceles
(c) equilateral
(d) obtuse angled isosceles

8. The points of z satisfying $\arg\left(\frac{z-1}{z+1}\right) = \frac{\pi}{4}$ lies on

(a) an arc of a circle
(b) line joining (1, 0), (–1, 0)
(c) pair of lines
(d) line joining (0, i), (0, $-i$)

9. If *a, b, c* and *u, v, w* are complex numbers representing vertices of two triangle such that $c = (1-r)a + rb$ and $w = (1-r)u + rv$, where *r* is a complex number, then the two triangles

(a) have same area
(b) are similar
(c) are congruent
(d) none of these

10. If $\alpha = \left(\cos\frac{8\pi}{11} + i\sin\frac{8\pi}{11}\right)$, then $\text{Re}(\alpha + \alpha^2 + \alpha^3 + \alpha^4 + \alpha^5)$ is equal to

(a) $\frac{1}{2}$
(b) $-\frac{1}{2}$
(c) 0
(d) none of these

11. The complex number *z* satisfies the condition $\left|z - \frac{25}{Z}\right| = 24$, then maximum value of $|z|$ is

(a) 25
(b) 30
(c) 32
(d) none of these

12. If $|z - 3i| = 3$ and $\arg z \in \left(0, \frac{\pi}{2}\right)$, then $\cot(\arg z) - \frac{6}{z}$ is equal to

(a) i
(b) $-i$
(c) $2i$
(d) $-2i$

13. If $\arg z < 0$, then $\arg(-z) - \arg(z)$ is equal to

(a) π
(b) $-\pi$
(c) $-\frac{\pi}{2}$
(d) $\frac{\pi}{2}$

14. If n_1, n_2 are positive integers, then $(1+i)^{n_1} + (1+i^3)^{n_1} + (1+i^5)^{n_2} + (1+i^7)^{n_2}$ is a real number if

(a) $n_1 = n_2$ only
(b) $n_1 + 1 = n_2$ only
(c) $n_1 = n_2 + 1$ only
(d) n_1, n_2 are positive integers

15. If $\log_{\left(\frac{1}{\sqrt{3}}\right)} \frac{|z|^2 - |z| + 1}{|z| + 2} > -2$, then the locus of *z* is

(a) the interior of a circle
(b) exterior of a circle
(c) boundary of a circle
(d) none of these

16. If z_1, z_2, z_3 are the vertices of an equilateral triangle (vertices taken anticlockwise) inscribed in the circle $|z| = 1$ and if $z_1 = i$, then

(a) $z_2 = -\frac{\sqrt{3}}{2} - \frac{i}{2}$
(b) $z_2 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$
(c) $z_3 = \frac{1}{2} - i\frac{\sqrt{3}}{2}$
(d) $z_3 = \frac{\sqrt{3}}{2} + \frac{i}{2}$

17. If $(1+\omega^2)^n = (1+\omega^4)^n$, where ω, ω^2 are non real cube roots of unity then least positive value of *n* is

(a) 2
(b) 3
(c) 5
(d) 6

18. If z_1 and z_2 are unimodular complex numbers such that $z_1^2 + z_2^2 = 4$, then the value of $(z_1 + \bar{z}_1)^2 + (z_2 + \bar{z}_2)^2$ is equal to
(a) 8 (b) 12
(c) 14 (d) none of these

19. If $|z_1| = 1, |z_2| = 2, |z_3| = 3$ and $|9z_1z_2 + 4z_1z_3 + z_2z_3| = 12$, then the value of $|z_1 + z_2 + z_3|$ is equal to
(a) 2 (b) 3
(c) 4 (d) 6

20. For the quadratic equation $az^2 + bz + c = 0$, where *a, b, c* are complex number the condition for both root real is
(a) $(b\bar{c} + c\bar{b})(a\bar{b} + \bar{a}b) + (c\bar{a} - \bar{a}c)^2 = 0$ (b) $(b\bar{c} - c\bar{b})(a\bar{b} - \bar{a}b) + (c\bar{a} - \bar{a}c)^2 = 0$
(c) $\frac{a}{\bar{a}} = -\frac{b}{\bar{b}} = \frac{c}{\bar{c}}$ (d) $\frac{a}{\bar{a}} = \frac{b}{\bar{b}} = \frac{c}{\bar{c}}$

21. If *z* satisfies $|z + 1 - i| \le 1$, then the value of *z* having least positive argument is
(a) $1 - i$ (b) $-1 + i$
(c) $-i$ (d) i

22. If $|z| = 1$ and $z \ne \pm 1$, then all the values of $\frac{z}{1 - z^2}$ lie on
(a) a line not passing through the origin (b) $|z| = \sqrt{2}$
(c) the *x*-axis (d) the *y*-axis

23. A man walks a distance of 3 units from the origin towards the north-east (*N* 45° *E*) direction. From there, he walks a distance of 4 units towards the north-west (*N* 45° *W*) direction to reach a point *P*. Then the position of *P* in the Argand plane is
(a) $3e^{i\pi/4} + 4i$ (b) $(3 - 4i)e^{i\pi/4}$
(c) $(4 + 3i)e^{i\pi/4}$ (d) $(3 + 4i)e^{i\pi/4}$

24. The value of $\tan\left[i \log \frac{a - ib}{a + ib}\right]$ is equal to
(a) $\frac{ab}{a^2 + b^2}$ (b) $\frac{2ab}{a^2 - b^2}$
(c) $\frac{ab}{a^2 - b^2}$ (d) $\frac{2ab}{a^2 + b^2}$

25. If z_1 and z_2 are any two point on the circle $|z| = 2$ such that $\arg \frac{z_1}{z_2} = \frac{\pi}{2}$, then locus of point of intersection of tangents at z_1 and z_2 is
(a) $|z| = 8$ (b) $|z| = 2\sqrt{2}$
(c) $|z| = 4$ (d) $|z| = 4\sqrt{2}$

EXERCISE – IV

ONE OR MORE THAN ONE CHOICE CORRECT

1. The complex number z satisfying $|z+\bar{z}|+|z-\bar{z}|=2$ and $|iz-1|+|z-i|=2$ is/are

(a) i (b) $-i$ (c) $\frac{1}{i}$ (d) $\frac{1}{i^3}$

2. If z satisfies $iz^2=\bar{z}^2+z$, then $\arg(z)$ is equal to (z is a non-zero complex number)

(a) $\frac{\pi}{4}$ (b) $\frac{3\pi}{4}$ (c) $-\frac{\pi}{4}$ (d) $-\frac{3\pi}{4}$

3. If $z=\frac{1+3i}{1+i}$, then

(a) $\text{Re}(z)=2\text{Im}(z)$ (b) $\text{Re}(z)+2\text{Im}(z)=0$

(c) $|z|=5$ (d) $\text{amp}(z)=\tan^{-1}2$

4. If $-3+ix^2y$ and x^2+y+4i are conjugate of each other then (x, y) is

(a) $(1,-4)$ (b) $(-1,-4)$

(c) $(2,1)$ (d) $(-2,1)$

5. If $g(x)$ and $h(x)$ are two polynomials such that $p(x)=g(x^3)+xh(x^3)$ and $p(x)$ is divisible by x^2+x+1, then

(a) $g(1)+h(1)=-1$ (b) $g(1)+h(1)=0$

(c) $g(1)-h(1)=0$ (d) $g(1)=h(1)=-1$

6. The equation whose roots are nth power of roots of equation $x^2-2x\cos\theta+1=0$ is given by

(a) $(x+\cos n\theta)^2+\sin^2 n\theta=0$ (b) $(x-\cos n\theta)^2+\sin^2 n\theta=0$

(c) $x^2-2x\cos n\theta+1=0$ (d) $x^2-2x\cos\theta+1=0$

7. If $\cos A+\cos B+\cos C=\sin A+\sin B+\sin C=0$, then

(a) $\sum\cos 2A=0$ (b) $\sum\sin 2A=0$

(c) $\sum\cos(A+B)=0$ (d) $\sum\sin(A+B)=0$

8. If $\dfrac{\tan\theta-i\left[\sin\frac{\theta}{2}+\cos\frac{\theta}{2}\right]}{1+2i\sin\left(\frac{\theta}{2}\right)}$ is purely imaginary if θ equals

(a) $n\pi+\frac{\pi}{4}$ (b) $n\pi-\frac{\pi}{4}$

(c) $2n\pi$ (d) $2n\pi+\frac{\pi}{4}$

9. If z_1, z_2, z_3, z_4 are roots of equation $a_0z^4 + a_1z^3 + a_2z^2 + a_3z + a_4 = 0$, where a_0, a_1, a_2, a_3, a_4 are real, then

(a) $\bar{z}_1, \bar{z}_2, \bar{z}_3, \bar{z}_4$ are also roots of equation

(b) z_1 is equal to any one of $\bar{z}_2, \bar{z}_3, \bar{z}_4$

(c) $-z_1, -z_2, -z_3, -z_4$ are also roots of equation

(d) $-\bar{z}_1, -\bar{z}_2, -\bar{z}_3, -\bar{z}_4$ are also roots of equation

10. If $amp(z_1z_2) = 0$ and $|z_1| = |z_2| = 1$, then

(a) $z_1 + z_2 = 0$ (b) $z_1z_2 = 1$

(c) $z_1 = \bar{z}_2$ (d) $z_1 - \bar{z}_2$ is purely real

11. If *ABCD* is a square (vertices taken in anticlockwise sense) and diagonals meets at origin. If vertex *A* represents the complex number *z*, then

(a) *B* represents iz (b) *D* represents $i\bar{z}$

(c) *C* represents $-z$ (d) *D* represents $-iz$

12. If $|z_1| = |z_2| = |z_3| = 1$, where z_1, z_2, z_3 represent vertices of equilateral triangle, then

(a) $z_1 + z_2 + z_3 = 0$ (b) $z_1z_2 + z_2z_3 + z_1z_3 = 0$

(c) $z_1z_2z_3 = 1$ (d) $z_1^2 + z_2^2 + z_3^2 = 0$

13. If $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2$, then

(a) $z_1\bar{z}_2 + z_2\bar{z}_1 = 0$ (b) $\frac{z_1}{z_2}$ is purely real

(c) $\frac{z_1}{z_2}$ is purely imaginary (d) $\text{amp}\left(\frac{z_1}{z_2}\right) = \pm\frac{\pi}{2}$

14. If z_1 and z_2 are two different complex number but $|z_1| = |z_2|$. If z_1 has positive real part and z_2 has negative imaginary part, then $\frac{z_1 + z_2}{z_1 - z_2}$ may be

(a) zero (b) real and positive

(c) real and negative (d) purely imaginary

15. The equation $z^6 + z^3 + 1 = 0$ has a root $re^{i\theta}$, where $0 < \theta < 90^o$, then the value of θ □is

(a) 40º (b) 80º

(c) 120º (d) 150º

EXERCISE – V

MATCH THE FOLLOWING

Note: Each statement in column – I has one or more than one match in column - II

1.

Column I	Column II
I. If $\arg\left(\frac{z-i}{z+i}\right)=\frac{3\pi}{4}$, then $\lvert z\rvert$ is always less than	**A.** $\frac{\pi}{4}$
II. If $\arg(z)=\frac{\pi}{4}$, then $\arg(z)-\arg\left(\frac{4}{z}\right)$ is equal to	**B.** 0
III. z_1 and z_2 are two complex numbers satisfying $\lvert z+2\rvert+\lvert z-2\rvert=4$ and $\lvert z\rvert=1$, then z_1+z_2 is equal to	**C.** 1
IV. Area of the region bounded by $\lvert z\rvert\le 1$ and $\frac{-\pi}{4}\le\arg(z)\le\frac{\pi}{4}$ is	**D.** $\frac{\pi}{2}$

Note: Each statement in column – I has one or more than one match in column - II

2.

Column I	Column II
I. If $\lvert z-2i\rvert+\lvert z-7i\rvert=k$, then locus of z is	**A.** ellipse if $k > 5$
II. If $\lvert z-1\rvert+\lvert z-6\rvert=k$, then locus of z is	**B.** hyperbola if $0 < k < 5$
III. If $\lvert\,\lvert z-3\rvert-\lvert z-4i\rvert\,\rvert=k$, then locus of z is	**C.** hyperbola if $k > 5$
IV. If $\lvert z-(2+4i)\rvert=\frac{k}{50}\lvert a\bar{z}+\bar{a}z+b\rvert$, where $a=3+4i$, then locus of z is	**D.** straight line if $k = 5$

Note: Each statement in column – I has one or more than one match in column - II

3.

Column I	Column II
I. If $\lvert z_1+z_2\rvert=\lvert z_1-z_2\rvert$	**A.** $\arg z_1=\arg z_2$
II. If $\lvert z_1+z_2\rvert=\lvert z_1\rvert+\lvert z_2\rvert$	**B.** $\arg\frac{z_1}{z_2}=\pm\frac{\pi}{2}$
III. If $\lvert z_1-z_2\rvert=\lvert\,\lvert z_1\rvert-\lvert z_2\rvert\,\rvert$	**C.** $\frac{z_1}{z_2}$ is purely real
IV. If $\lvert z_1-z_2\rvert=\lvert z_1\rvert+\lvert z_2\rvert$	**D.** $\frac{z_1}{z_2}$ is purely imaginary

REASONING TYPE

Directions: Read the following questions and choose

(A) **If both the statements are true and statement-2** is the correct explanation of **statement-1.**

(B) **If both the statements are true but statement-2** is not the correct explanation of **statement-1.**

(C) **If statement-1** is True and **statement-2** is False.

(D) **If statement-1** is False and **statement-2** is True.

1. **Statement-1:** A triangle is formed by joining the points $A(z_1)$, $B(z_2)$ and $C(z_3)$. If anglular bisector of $\angle A$ meets the circumcircle at *P*, then *P* is represented by the complex number $\sqrt{z_2 z_3}$.

Statement-2: Angle subtended by chord at the centre of the circle is double the angle subtended on the circumference of the circle.

(a) A (b) B (c) C (d) D

2. **Statement-1:** If $z_1^2 + z_2^2 + z_1 z_2 = 0$, then the points represented by z_1, z_2 and origin form vertices of an isosceles triangles. (where z_1, z_2 are non zero complex numbers)

Statement-2: $z_2 = z_1 e^{\frac{i\,2\pi}{3}}$.

(a) A (b) B (c) C (d) D

3. **Statement-1:** Sum of square of four non-zero complex numbers may be zero.

Statement-2: z_1, z_2, z_3, z_4 may be the fourth roots of unity.

(a) A (b) B (c) C (d) D

4. **Statement-1:** If $1 + ixy$ and $y + 2i$ are conjugate to each other then $x^2 + y^2 = 5$.

Statement-2: If sum and product of two complex numbers is real, then they are conjugate complex numbers.

(a) A (b) B (c) C (d) D

5. **Statement-1:** A complex number *z* is such that *z* > 2, then the locus of *z* is exterior of a circle whose radius is 2.

Statement-2: The only order relation, that can exist in complex numbers is equality

(a) A (b) B (c) C (d) D

LINKED COMPREHENSION TYPE

The equation of a circle in argand plane can be written in several ways. For example, the following relations:

(i) $|z - 2i| = 2$ (ii) $\left|\frac{z-2i}{z+2i}\right| = 3$

(iii) $\text{Re}\left(\frac{z-2}{z+4i}\right) = 0$ (iv) $|z - z_1|^2 + |z - z_2|^2 = k$ (where $|z_1 - z_2|^2 < 2k$)

define a circle. We can get the cartesian equation of the circle by putting $z = x + iy$, $x, y, \in R$. Circle can be defined through a parameter also. For example, the relation $z = \frac{2i - t}{3 + it}(t \in R)$ defines a circle. If we want its cartesian equation, we must find real part *x* and imaginary party *y* of *z* in terms of *t*. The eliminant will yield the equation of the circle.

1. The centre of the circle $z = \frac{3i - t}{2 + it}(t \in R)$ must be

(a) $\left(0, \frac{3}{4}\right)$ (b) $\left(0, \frac{5}{4}\right)$

(c) $\left(0, \frac{2}{3}\right)$ (d) $(0, 0)$

2. The radius of the circle $z = \frac{3i - t}{2 + it}$ must be

(a) $\frac{1}{2}$

(b) $\frac{1}{4}$

(c) $\frac{1}{\sqrt{2}}$

(d) 1

3. The centre of the circle $\left|\frac{z - z_1}{z - z_2}\right| = k;\ k \neq 1$ is

(a) $\frac{k^2 z_2 + z_1}{k^2 + 1}$

(b) $\frac{k^2 z_1 + z_2}{k^2 + 1}$

(c) $\frac{k^2 z_2 - z_1}{k^2 - 1}$

(d) $\frac{k^2 z_1 - z_2}{k^2 - 1}$

EXERCISE – VI

SUBJECTIVE PROBLEMS

1. Find the complex number Z which simultaneously satisfies the equation
$\left|\frac{Z-12}{Z-8i}\right| = 5/3; \left|\frac{Z-4}{Z-8}\right| = 1.$

2. **(i)** If ω is a complex cube root of unity, prove that
$(a + b\omega + c\omega^2)^3 + (a + b\omega^2 + c\omega)^3 = (2a - b - c)(2b - a - c)(2c - a - b)$.

(ii) Find the value of $\sum_{r=1}^{4} \frac{1}{2-\alpha^r}$, where α^k ($k = 0, 1, 2, 3, 4$) are fifth roots of unity.

3. If $(a+\omega)^{-1} + (b+\omega)^{-1} + (c+\omega)^{-1} + (d+\omega)^{-1} = 2\omega^{-1}$,
$(a+\omega')^{-1} + (b+\omega')^{-1} + (c+\omega')^{-1} + (d+\omega')^{-1} = 2(\omega')^{-1}$, where ω and ω' are the imaginary cube roots of unity, prove that $(a+1)^{-1} + (b+1)^{-1} + (c+1)^{-1} + (d+1)^{-1} = 2$.

4. **(i)** Let A and B be two complex numbers such that $\frac{A}{B} + \frac{B}{A} = 1$, then prove that the origin and the two points represented by A and B form vertices of an equilateral triangle.

(ii) Find radius of arc given by locus of z if $\arg\left(\frac{z-4i}{z-3}\right) = \frac{\pi}{3}$.

(iii) Let Z_1 and Z_2 be the roots of $Z^2 + pZ + q = 0$ where the coefficients p and q may be complex numbers. Let A and B represent Z_1 and Z_2 in the complex plane. If $\underline{|AOB} = \alpha \neq 0$ and $OA = OB$, where O is the origin, prove that $p^2 = 4q\cos^2\left(\frac{\alpha}{2}\right)$.

5. A cubic equation $f(x) = 0$ has one real root α and two complex roots $\beta \pm i\gamma$. Points A, B and C represent roots α, $\beta + i\gamma$, $\beta - i\gamma$ respectively on the argand diagram. Show that the roots of the derived equation $f'(x) = 0$ are complex if A falls inside one of the two equilateral triangle described on base BC.

6. **(i)** Show that the roots of the equation $Z^n = (Z + 1)^n$ when represented on the Argand diagram are collinear points.

(ii) Find the point in Argand plane which is equidistant from roots of $(z+1)^4 = 16z^4$.

7. Two different non-parallel lines meet the circle $|z| = r$ in the point a, b, c and d respectively. Prove that these lines meet at a point Z given by $z = \frac{a^{-1} + b^{-1} - c^{-1} - d^{-1}}{a^{-1}b^{-1} - c^{-1}d^{-1}}$.

8. Examine the location in the Argand's diagram of the point represented by the roots of the equation $Z^n \cos\theta_0 + Z^{n-1}\cos\theta_1 + ... + \cos\theta_n = 2$ where $\theta_0, \theta_1, ..., \theta_n$ are real angles and $n \geq 2$.

9. Assume that A_i ($i = 1, 2, 3, …, n$) are the vertices of a regular polygon of n sides inscribed in a circle of radius unity. Show that

(i) $|A_1A_2|^2 + |A_1A_3|^2 + ... + |A_1A_n|^2 = 2n$.   (ii) $|A_1A_2| \cdot |A_1A_3| ..... |A_1A_n| = n$.

10. A, B, C are the points representing the complex numbers z_1, z_2, z_3 respectively on the complex plane and the circumcentre of the triangle ABC lies at the origin. If the altitude of the triangle through the vertex A meets the circumcircle again at P, then prove that P represents the complex number $-z_2z_3/z_1$.

ANSWERS

EXERCISE – I

CBSE PROBLEMS

1. -1

5. (i) $\sec\alpha\,(\cos\alpha + i\sin\alpha)$

(ii) $r\,(\cos\theta + i\sin\theta)$, $r = \sqrt{2}\left(\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}\right)$, $\theta = \frac{\pi}{4} + \frac{\alpha}{2}$.

6. (i) $\pm(4-3i)$ (ii) $\pm(2+3i)$ (iii) $\pm[(a+b) - i\,(a-b)]$

(iv) $\pm\left(x - \frac{1}{x} + 2i\right)$

7. (i) $x = \frac{5}{13}, y = \frac{14}{13}$ (ii) $x = 3, y = -1$

8. (i) $1 + 2\sqrt{2}\,i$ (ii) -4

10. $A^2 + B^2$

11. $\pm 2(1 \pm i)$

12. $\theta = n\pi$

13. $\frac{63}{25} + \frac{16}{25}i$

14. (i) $\frac{1}{\sqrt{2}}$ (ii) 2

15. (i) $\frac{\pi}{2}$ (ii) $-\frac{5\pi}{6}$

16. (i) $\frac{1}{\sqrt{2}}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$ (ii) $\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

17. 1

18. 12

19. $(1-ab)^2 + (a+b)^2$

20. 2

EXERCISE – II

NEET-SINGLE CHOICE CORRECT

1. (c)	2. (c)	3. (b)	4. (a)	5. (c)
6. (b)	7. (a)	8. (b)	9. (a)	10. (a)
11. (d)	12. (d)	13. (b)	14. (a)	15. (a)
16. (b)	17. (d)	18. (a)	19. (a)	20. (a)
21. (a)	22. (c)	23. (a)	24. (b)	25. (a)

EXERCISE – III

IIT-JEE – SINGLE CHOICE CORRECT

1. (b)	2. (a)	3. (a)	4. (d)	5. (c)
6. (d)	7. (c)	8. (a)	9. (b)	10. (b)
11. (a)	12. (a)	13. (a)	14. (d)	15. (a)
16. (a)	17. (b)	18. (b)	19. (a)	20. (d)
21. (d)	22. (d)	23. (d)	24. (b)	25. (b)

EXERCISE – IV

ONE OR MORE THAN ONE CHOICE CORRECT

1. (a, b, c, d)	2. (c)	3. (a, c)	4. (a, b)	5. (b, c)
6. (b, c)	7. (a, b,. c, d)	8. (a, c, d)	9. (a, b)	10. (b, c, d)
11. (a, c, d)	12. (a, b, d)	13. (a, c, d)	14. (a, d)	15.(a, b)

EXERCISE – V

MATCH THE FOLLOWING

1. I-(C); II-(D); III-(B); IV-(A)

2. I-(A), (D); II-(A), (D); III-(B), (D); IV-(C)

3. I-(B), (D); II-(A), (C); III-(A), (C); IV-(C)

REASONING TYPE

1. (a)	2. (a)	3. (a)	4. (a)	5. (d)

LINKED COMPREHENSION TYPE

1. (b)	2. (b)	3. (c)

EXERCISE – VI

SUBJECTIVE PROBLEMS

1. $Z = 6 + 8i \text{ or } 6 + 17i$

2. (ii) $\frac{49}{31}$

4. (ii) $\frac{5}{\sqrt{3}}$

6. (ii) $\left(\frac{1}{3}, 0\right)$

8. Roots are located outside $|Z| = \frac{1}{2}$